Infinite generation of non-cocompact lattices on right-angled buildings

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Let Γ be a non-cocompact lattice on a locally finite regular right-angled building X. We prove that if Γ has a strict fundamental domain then Γ is not finitely generated. We use the separation properties of subcomplexes of X called tree-walls.

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Tree lattices have been well-studied (see [BL]). Less understood are lattices on higher-dimensional CAT(0) complexes. In this paper, we consider lattices on X a locally finite, regular right-angled building (see Davis [D] and Section 1 below). Examples of such X include products of locally finite regular or biregular trees, or Bourdon's building $I_{p,q}$ [B], which has apartments hyperbolic planes tesselated by right-angled p–gons and all vertex links the complete bipartite graph $K_{q,q}$.

Let G be a closed, cocompact group of type-preserving automorphisms of X, equipped with the compact-open topology, and let Γ be a lattice in G. That is, Γ is discrete and the series $\sum |\operatorname{Stab}_\Gamma(\phi)|^{-1}$ converges, where the sum is over the set of chambers ϕ of a fundamental domain for Γ. The lattice Γ is cocompact in G if and only if the quotient $\Gamma\backslash X$ is compact.

If there is a subcomplex $Y \subset X$ containing exactly one point from each Γ–orbit on X, then Y is called a *strict fundamental domain* for Γ. Equivalently, Γ has a strict fundamental domain if $\Gamma\backslash X$ may be embedded in X.

Any cocompact lattice in G is finitely generated. We prove:

Theorem 1 *Let Γ be a non-cocompact lattice in G. If Γ has a strict fundamental domain, then Γ is* not *finitely generated.*

We note that Theorem 1 contrasts with the finite generation of lattices on many buildings whose chambers are simplices. Results of, for example, Ballmann–Świątkowski [BŚ], Dymara–Januszkiewicz [DJ], and Zuk [Zu], establish that all lattices on many such buildings have Kazhdan's Property (T). Hence by a well-known result due to Kazhdan [K], these lattices are finitely generated.

Our proof of Theorem 1, in Section 3 below, uses the separation properties of subcomplexes of X which we call *tree-walls*. These generalize the tree-walls (in French, *arbre-murs*) of $I_{p,q}$, which were introduced by Bourdon in [B]. We define tree-walls and establish their properties in Section 2 below.

The following examples of non-cocompact lattices on right-angled buildings are known to us.

(1) For $i = 1, 2$, let G_i be a rank one Lie group over a nonarchimedean locally compact field whose Bruhat–Tits building is the locally finite regular or biregular tree T_i. Then any irreducible lattice in $G = G_1 \times G_2$ is finitely generated (Raghunathan [Ra]). Hence by Theorem 1 above, such lattices on $X = T_1 \times T_2$ cannot have strict fundamental domain.

(2) Let Λ be a minimal Kac–Moody group over a finite field $\mathbb{F}_q$ with right-angled Weyl group W. Then Λ has locally finite, regular right-angled twin buildings $X_+ \cong X_-$, and Λ acts diagonally on the product $X_+ \times X_-$. For q large enough:

  (a) By Theorem 0.2 of Carbone–Garland [CG] or Theorem 1(i) of Rémy [Ré], the stabilizer in Λ of a point in X_- is a non-cocompact lattice in Aut(X_+). Any such lattice is contained in a negative maximal spherical parabolic subgroup of Λ, which has strict fundamental domain a sector in X_+, and so any such lattice has strict fundamental domain.

  (b) By Theorem 1(ii) of Rémy [Ré], the group Λ is itself a non-cocompact lattice in Aut(X_+) $\times$ Aut(X_-). Since Λ is finitely generated, Theorem 1 above implies that Λ does not have strict fundamental domain in $X = X_+ \times X_-$.

  (c) By Section 7.3 of Gramlich–Horn–Mühlherr [GHM], the fixed set G_θ of certain involutions θ of Λ is a lattice in Aut(X_+), which is sometimes cocompact and sometimes non-cocompact. Moreover, by [GHM, Remark 7.13], there exists θ such that G_θ is not finitely generated.

(3) In [T], the first author constructed a functor from graphs of groups to complexes of groups, which extends the corresponding tree lattice to a lattice in Aut(X) where X is a regular right-angled building. The resulting lattice in Aut(X) has strict fundamental domain if and only if the original tree lattice has strict fundamental domain.

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1 Right-angled buildings

In this section we recall the basic definitions and some examples for right-angled buildings. We mostly follow Davis [D], in particular Section 12.2 and Example 18.1.10. See also [KT, Sections 1.2–1.4].

Let (W, S) be a right-angled Coxeter system. That is,

$$W = \langle S \mid (st)^{m_{st}} = 1 \rangle$$

where $m_{ss} = 1$ for all $s \in S$, and $m_{st} \in \{2, \infty\}$ for all $s, t \in S$ with $s \neq t$. We will discuss the following examples:

- $W_1 = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong D_\infty$, the infinite dihedral group;
- $W_2 = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^2 = 1 \rangle \cong (C_2 \times C_2) * C_2$, where C_2 is the cyclic group of order 2; and
- The Coxeter group W_3 generated by the set of reflections S in the sides of a right-angled hyperbolic p–gon, $p \geq 5$. That is,

  $$W_3 = \langle s_1, \ldots, s_p \mid s_i^2 = (s_i s_{i+1})^2 = 1 \rangle$$

  with cyclic indexing.

Fix $(q_s)_{s \in S}$ a family of integers with $q_s \geq 2$. Given any family of groups $(H_s)_{s \in S}$ with $|H_s| = q_s$, let H be the quotient of the free product of the $(H_s)_{s \in S}$ by the normal subgroup generated by the commutators $\{[h_s, h_t] : h_s \in H_s, h_t \in H_t, m_{st} = 2\}$.

Now let X be the piecewise Euclidean CAT(0) geometric realization of the chamber system $\Phi = \Phi\left(H, \{1\}, (H_s)_{s \in S}\right)$. Then X is a locally finite, regular right-angled building, with chamber set $\mathrm{Ch}(X)$ in bijection with the elements of the group H. Let $\delta_W \colon \mathrm{Ch}(X) \times \mathrm{Ch}(X) \to W$ be the W–valued distance function and let $l_S \colon W \to \mathbb{N}$ be word length with respect to the generating set S. Denote by $d_W \colon \mathrm{Ch}(X) \times \mathrm{Ch}(X) \to \mathbb{N}$

the *gallery distance* $l_S \circ \delta_W$. That is, for two chambers ϕ and ϕ' of X, $d_W(\phi, \phi')$ is the length of a minimal gallery from ϕ to ϕ'.

Suppose that ϕ and ϕ' are s–adjacent chambers, for some $s \in S$. That is, $\delta_W(\phi, \phi') = s$. The intersection $\phi \cap \phi'$ is called an *s–panel*. By definition, since X is regular, each s–panel is contained in q_s distinct chambers. For distinct $s, t \in S$, the s–panel and t–panel of any chamber ϕ of X have nonempty intersection if and only if $m_{st} = 2$. Each s–panel of X is reduced to a vertex if and only if $m_{st} = \infty$ for all $t \in S - \{s\}$.

For the examples W_1, W_2, and W_3 above, respectively:

- The building X_1 is a tree with each chamber an edge, each s–panel a vertex of valence q_s, and each t–panel a vertex of valence q_t. That is, X_1 is the (q_s, q_t)–biregular tree. The apartments of X_1 are bi-infinite rays in this tree.

- The building X_2 has chambers and apartments as shown in Figure 1 below. The r– and s–panels are 1–dimensional and the t–panels are vertices.

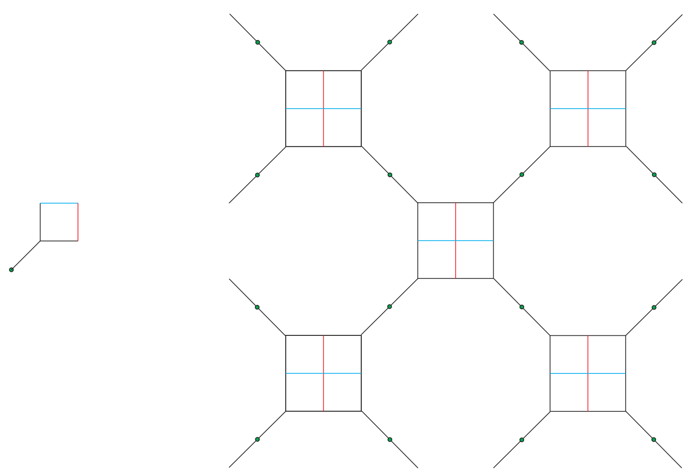

Figure 1: A chamber (on the left) and part of an apartment (on the right) for the building X_2.

- The building X_3 has chambers p–gons and s–panels the edges of these p–gons. If $q_s = q \geq 2$ for all $s \in S$, then each s–panel is contained in q chambers, and X_3, equipped with the obvious piecewise hyperbolic metric, is Bourdon's building $I_{p,q}$.

2 Tree-walls

We now generalize the notion of tree-wall due to Bourdon [B]. We will use basic facts about buildings, found in, for example, Davis [D]. Our main results concerning tree-walls are Corollary 3 below, which describes three possibilities for tree-walls, and Proposition 6 below, which generalizes the separation property 2.4.A(ii) of [B].

Let X be as in Section 1 above and let $s \in S$. As in [B, Section 2.4.A], we define two s–panels of X to be *equivalent* if they are contained in a common wall of type s in some apartment of X. A *tree-wall of type* s is then an equivalence class under this relation. We note that in order for walls and thus tree-walls to have a well-defined type, it is necessary only that all finite m_{st}, for $s \neq t$, be even. Tree-walls could thus be defined for buildings of type any even Coxeter system, and they would have properties similar to those below. We will however only explicitly consider the right-angled case.

Let $\mathcal{T}$ be a tree-wall of X, of type s. We define a chamber ϕ of X to be *epicormic at* $\mathcal{T}$ if the s–panel of ϕ is contained in $\mathcal{T}$, and we say that a gallery $\alpha = (\phi_0, \ldots, \phi_n)$ *crosses* $\mathcal{T}$ if, for some $0 \leq i < n$, the chambers ϕ_i and ϕ_{i+1} are epicormic at $\mathcal{T}$.

By the definition of tree-wall, if $\phi \in \mathrm{Ch}(X)$ is epicormic at $\mathcal{T}$ and $\phi' \in \mathrm{Ch}(X)$ is t–adjacent to ϕ with $t \neq s$, then ϕ' is epicormic at $\mathcal{T}$ if and only if $m_{st} = 2$. Let $s^\perp := \{t \in S \mid m_{st} = 2\}$ and denote by $\langle s^\perp \rangle$ the subgroup of W generated by the elements of $s^\perp$. If $s^\perp$ is empty then by convention, $\langle s^\perp \rangle$ is trivial. For the examples in Section 1 above:

- in W_1, both $\langle s^\perp \rangle$ and $\langle t^\perp \rangle$ are trivial;
- in W_2, $\langle r^\perp \rangle = \langle s \rangle \cong C_2$ and $\langle s^\perp \rangle = \langle r \rangle \cong C_2$, while $\langle t^\perp \rangle$ is trivial; and
- in W_3, $\langle s_i^\perp \rangle = \langle s_{i-1}, s_{i+1} \rangle \cong D_\infty$ for each $1 \leq i \leq p$.

Lemma 2 *Let $\mathcal{T}$ be a tree-wall of X of type s. Let ϕ be a chamber which is epicormic at $\mathcal{T}$ and let A be any apartment containing ϕ.*

(1) *The intersection $\mathcal{T} \cap A$ is a wall of A, hence separates A.*

(2) *There is a bijection between the elements of the group $\langle s^\perp \rangle$ and the set of chambers of A which are epicormic at $\mathcal{T}$ and in the same component of $A - \mathcal{T} \cap A$ as ϕ.*

Proof Part (1) is immediate from the definition of tree-wall. For Part (2), let $w \in \langle s^\perp \rangle$ and let $\psi = \psi_w$ be the unique chamber of A such that $\delta_W(\phi, \psi) = w$. We claim that ψ is epicormic at $\mathcal{T}$ and in the same component of $A - \mathcal{T} \cap A$ as ϕ.

For this, let $s_1 \cdots s_n$ be a reduced expression for w and let $\alpha = (\phi_0, \ldots, \phi_n)$ be the minimal gallery from $\phi = \phi_0$ to $\psi = \phi_n$ of type $(s_1, \ldots, s_n)$. Since w is in $\langle s^{\perp} \rangle$, we have $m_{s_i s} = 2$ for $1 \leq i \leq n$. Hence by induction each ϕ_i is epicormic at $\mathcal{T}$, and so $\psi = \phi_n$ is epicormic at $\mathcal{T}$. Moreover, since none of the s_i are equal to s, the gallery α does not cross $\mathcal{T}$. Thus $\psi = \psi_w$ is in the same component of $A - \mathcal{T} \cap A$ as ϕ.

It follows that $w \mapsto \psi_w$ is a well-defined, injective map from $\langle s^{\perp} \rangle$ to the set of chambers of A which are epicormic at $\mathcal{T}$ and in the same component of $A - \mathcal{T} \cap A$ as ϕ. To complete the proof, we will show that this map is surjective. So let ψ be a chamber of A which is epicormic at $\mathcal{T}$ and in the same component of $A - \mathcal{T} \cap A$ as ϕ, and let $w = \delta_W(\phi, \psi)$.

If $\langle s^{\perp} \rangle$ is trivial then $\psi = \phi$ and $w = 1$, and we are done. Next suppose that the chambers ϕ and ψ are t–adjacent, for some $t \in S$. Since both ϕ and ψ are epicormic at $\mathcal{T}$, either $t = s$ or $m_{st} = 2$. But ψ is in the same component of $A - \mathcal{T} \cap A$ as ϕ, so $t \neq s$, hence $w = t$ is in $\langle s^{\perp} \rangle$ as required. If $\langle s^{\perp} \rangle$ is finite, then finitely many applications of this argument will finish the proof. If $\langle s^{\perp} \rangle$ is infinite, we have established the base case of an induction on $n = l_S(w)$.

For the inductive step, let $s_1 \cdots s_n$ be a reduced expression for w and let $\alpha = (\phi_0, \ldots, \phi_n)$ be the minimal gallery from $\phi = \phi_0$ to $\psi = \phi_n$ of type $(s_1, \ldots, s_n)$. Since ϕ and ψ are in the same component of $A - \mathcal{T} \cap A$ and α is minimal, the gallery α does not cross $\mathcal{T}$. We claim that s_n is in $s^{\perp}$. First note that $s_n \neq s$ since α does not cross $\mathcal{T}$ and $\psi = \phi_n$ is epicormic at $\mathcal{T}$. Now denote by $\mathcal{T}_n$ the tree-wall of X containing the s_n–panel $\phi_{n-1} \cap \phi_n$. Since α is minimal and crosses $\mathcal{T}_n$, the chambers $\phi = \phi_0$ and $\psi = \phi_n$ are separated by the wall $\mathcal{T}_n \cap A$. Thus the s–panel of ϕ and the s–panel of ψ are separated by $\mathcal{T}_n \cap A$. As the s–panels of both ϕ and ψ are in the wall $\mathcal{T} \cap A$, it follows that the walls $\mathcal{T}_n \cap A$ and $\mathcal{T} \cap A$ intersect. Hence $m_{s_n s} = 2$, as claimed.

Now let $w' = ws_n = s_1 \cdots s_{n-1}$ and let ψ' be the unique chamber of A such that $\delta_W(\phi, \psi') = w'$. Since s_n is in $s^{\perp}$ and ψ' is s_n–adjacent to ψ, the chamber ψ' is epicormic at $\mathcal{T}$ and in the same component of $A - \mathcal{T} \cap A$ as ϕ. Moreover $s_1 \cdots s_{n-1}$ is a reduced expression for w', so $l_S(w') = n - 1$. Hence by the inductive assumption, w' is in $\langle s^{\perp} \rangle$. Therefore $w = w' s_n$ is in $\langle s^{\perp} \rangle$, which completes the proof. □

Corollary 3 *The following possibilities for tree-walls in X may occur.*

(1) *Every tree-wall of type s is reduced to a vertex if and only if $\langle s^{\perp} \rangle$ is trivial.*

(2) *Every tree-wall of type s is finite but not reduced to a vertex if and only if $\langle s^{\perp} \rangle$ is finite but nontrivial.*

(3) *Every tree-wall of type s is infinite if and only if $\langle s^{\perp}\rangle$ is infinite.*

Proof Let $\mathcal{T}$, ϕ, and A be as in Lemma 2 above. The set of s–panels in the wall $\mathcal{T}\cap A$ is in bijection with the set of chambers of A which are epicormic at $\mathcal{T}$ and in the same component of $A-\mathcal{T}\cap A$ as ϕ. □

For the examples in Section 1 above:

- in X_1, every tree-wall of type s and of type t is a vertex;
- in X_2, the tree-walls of types both r and s are finite and 1–dimensional, while every tree-wall of type t is a vertex; and
- in X_3, all tree-walls are infinite, and are 1–dimensional.

Corollary 4 *Let $\mathcal{T}$, ϕ, and A be as in Lemma 2 above and let*

$$\rho=\rho_{\phi,A}\colon\ X\to A$$

be the retraction onto A centered at ϕ. Then $\rho^{-1}(\mathcal{T}\cap A)=\mathcal{T}$.

Proof Let ψ be any chamber of A which is epicormic at $\mathcal{T}$ and is in the same component of $A-\mathcal{T}\cap A$ as ϕ. Then by the proof of Lemma 2 above, $w:=\delta_W(\phi,\psi)$ is in $\langle s^{\perp}\rangle$. Let ψ' be a chamber in the preimage $\rho^{-1}(\psi)$ and let A' be an apartment containing both ϕ and ψ'. Since the retraction ρ preserves W–distances from ϕ, we have that $\delta_W(\phi,\psi')=w$ is in $\langle s^{\perp}\rangle$. Again by the proof of Lemma 2, it follows that the chamber ψ' is epicormic at $\mathcal{T}$. But the image under ρ of the s–panel of ψ' is the s–panel of ψ. Thus $\rho^{-1}(\mathcal{T}\cap A)=\mathcal{T}$, as required. □

Lemma 5 *Let $\mathcal{T}$ be a tree-wall and let ϕ and ϕ' be two chambers of X. Let α be a minimal gallery from ϕ to ϕ' and let β be any gallery from ϕ to ϕ'. If α crosses $\mathcal{T}$ then β crosses $\mathcal{T}$.*

Proof Suppose that α crosses $\mathcal{T}$. Since α is minimal, there is an apartment A of X which contains α, and hence the wall $\mathcal{T}\cap A$ separates ϕ from ϕ'. Choose a chamber ϕ_0 of A which is epicormic at $\mathcal{T}$ and consider the retraction $\rho=\rho_{\phi_0,A}$ onto A centered at ϕ_0. Since ϕ and ϕ' are in A, ρ fixes ϕ and ϕ'. Hence $\rho(\beta)$ is a gallery in A from ϕ to ϕ', and so $\rho(\beta)$ crosses $\mathcal{T}\cap A$. By Corollary 4 above, $\rho^{-1}(\mathcal{T}\cap A)=\mathcal{T}$. Therefore β crosses $\mathcal{T}$. □

Proposition 6 *Let $\mathcal{T}$ be a tree-wall of type s. Then $\mathcal{T}$ separates X into q_s gallery-connected components.*

Proof Fix an s–panel in $\mathcal{T}$ and let $\phi_1, \ldots, \phi_{q_s}$ be the q_s chambers containing this panel. Then for all $1 \leq i < j \leq q_s$, the minimal gallery from ϕ_i to ϕ_j is just (ϕ_i, ϕ_j), and hence crosses $\mathcal{T}$. Thus by Lemma 5 above, any gallery from ϕ_i to ϕ_j crosses $\mathcal{T}$. So the q_s chambers $\phi_1, \ldots, \phi_{q_s}$ lie in q_s distinct components of $X - \mathcal{T}$.

To complete the proof, we show that $\mathcal{T}$ separates X into at most q_s components. Let ϕ be any chamber of X. Then among the chambers $\phi_1, \ldots, \phi_{q_s}$, there is a unique chamber, say ϕ_1, at minimal gallery distance from ϕ. It suffices to show that ϕ and ϕ_1 are in the same component of $X - \mathcal{T}$.

Let α be a minimal gallery from ϕ to ϕ_1 and let A be an apartment containing α. Then there is a unique chamber of A which is s–adjacent to ϕ_1. Hence A contains ϕ_i for some $i > 1$, and the wall $\mathcal{T} \cap A$ separates ϕ_1 from ϕ_i. Since α is minimal and $d_W(\phi, \phi_1) < d_W(\phi, \phi_i)$, the Exchange Condition (see [D, page 35]) implies that a minimal gallery from ϕ to ϕ_i may be obtained by concatenating α with the gallery (ϕ_1, ϕ_i). Since a minimal gallery can cross $\mathcal{T} \cap A$ at most once, α does not cross $\mathcal{T} \cap A$. Thus ϕ and ϕ_1 are in the same component of $X - \mathcal{T}$, as required. □

3 Proof of Theorem

Let G be as in the introduction and let Γ be a non-cocompact lattice in G with strict fundamental domain. Fix a chamber ϕ_0 of X. For each integer $n \geq 0$ define

$$D(n) := \{\, \phi \in \mathrm{Ch}(X) \mid d_W(\phi, \Gamma\phi_0) \leq n \,\}.$$

Then $D(0) = \Gamma\phi_0$, and for every $n > 0$ every connected component of $D(n)$ contains a chamber in $\Gamma\phi_0$. To prove Theorem 1, we will show that there is no $n > 0$ such that $D(n)$ is connected.

Let Y be a strict fundamental domain for Γ which contains ϕ_0. For each chamber ϕ of X, denote by ϕ_Y the representative of ϕ in Y.

Lemma 7 *Let ϕ and ϕ' be t–adjacent chambers in X, for $t \in S$. Then either $\phi_Y = \phi'_Y$, or ϕ_Y and ϕ'_Y are t–adjacent.*

Proof It suffices to show that the t–panel of ϕ_Y is the t–panel of ϕ'_Y. Since Y is a subcomplex of X, the t–panel of ϕ_Y is contained in Y. By definition of a strict fundamental domain, there is exactly one representative in Y of the t–panel of ϕ. Hence the unique representative in Y of the t–panel of ϕ is the t–panel of ϕ_Y. Similarly, the unique representative in Y of the t–panel of ϕ' is the t–panel of ϕ'_Y. But ϕ and ϕ' are

t–adjacent, hence have the same t–panel, and so it follows that ϕ_Y and ϕ'_Y have the same t–panel. □

Corollary 8 *The fundamental domain Y is gallery-connected.*

Lemma 9 *For all $n > 0$, the fundamental domain Y contains a pair of adjacent chambers ϕ_n and ϕ'_n such that, if $\mathcal{T}_n$ denotes the tree-wall separating ϕ_n from ϕ'_n:*

(1) *the chambers ϕ_0 and ϕ_n are in the same gallery-connected component of $Y - \mathcal{T}_n \cap Y$;*

(2) $\min\{d_W(\phi_0, \phi) \mid \phi \in \mathrm{Ch}(X)$ *is epicormic at* $\mathcal{T}_n\} > n$*; and*

(3) *there is a $\gamma \in \mathrm{Stab}_\Gamma(\phi'_n)$ which does not fix ϕ_n.*

Proof Fix $n > 0$. Since Γ is not cocompact, Y is not compact. Thus there exists a tree-wall $\mathcal{T}_n$ with $\mathcal{T}_n \cap Y$ nonempty such that for every $\phi \in \mathrm{Ch}(X)$ which is epicormic at $\mathcal{T}_n$, $d_W(\phi_0, \phi) > n$. Let s_n be the type of the tree-wall $\mathcal{T}_n$. Then by Corollary 8 above, there is a chamber ϕ_n of Y which is epicormic at $\mathcal{T}_n$ and in the same gallery-connected component of $Y - \mathcal{T}_n \cap Y$ as ϕ_0, such that for some chamber ϕ'_n which is s_n–adjacent to ϕ_n, ϕ'_n is also in Y. Now, as Γ is a non-cocompact lattice, the orders of the Γ–stabilizers of the chambers in Y are unbounded. Hence the tree-wall $\mathcal{T}_n$ and chambers ϕ_n and ϕ'_n may be chosen so that $|\,\mathrm{Stab}_\Gamma(\phi_n)| < |\,\mathrm{Stab}_\Gamma(\phi'_n)|$. □

Let ϕ_n, ϕ'_n, $\mathcal{T}_n$, and γ be as in Lemma 9 above and let $s = s_n$ be the type of the tree-wall $\mathcal{T}_n$. Let α be a gallery in $Y - \mathcal{T}_n \cap Y$ from ϕ_0 to ϕ_n. The chambers ϕ_n and $\gamma \cdot \phi_n$ are in two distinct components of $X - \mathcal{T}_n$, since they both contain the s–panel $\phi_n \cap \phi'_n \subseteq \mathcal{T}_n$, which is fixed by γ. Hence the galleries α and $\gamma \cdot \alpha$ are in two distinct components of $X - \mathcal{T}_n$, and so the chambers ϕ_0 and $\gamma \cdot \phi_0$ are in two distinct components of $X - \mathcal{T}_n$. Denote by X_0 the component of $X - \mathcal{T}_n$ which contains ϕ_0, and put $Y_0 = Y \cap X_0$.

Lemma 10 *Let ϕ be a chamber in X_0 that is epicormic at $\mathcal{T}_n$. Then ϕ_Y is in Y_0 and is epicormic at $\mathcal{T}_n \cap Y$.*

Proof We consider three cases, corresponding to the possibilities for tree-walls in Corollary 3 above.

(1) If $\mathcal{T}_n$ is reduced to a vertex, there is only one chamber in X_0 which is epicormic at $\mathcal{T}_n$, namely ϕ_n. Thus $\phi = \phi_n = \phi_Y$ and we are done.

(2) If $\mathcal{T}_n$ is finite but not reduced to a vertex, the result follows by finitely many applications of Lemma 7 above.

(3) If $\mathcal{T}_n$ is infinite, the result follows by induction, using Lemma 7 above, on

$$k := \min\{d_W(\phi, \psi) \mid \psi \text{ is a chamber of } Y_0 \text{ epicormic at } \mathcal{T}_n \cap Y\}. \qquad \square$$

Lemma 11 *For all $n > 0$, the complex $D(n)$ is not connected.*

Proof Fix $n > 0$, and let α be a gallery in X between a chamber in $X_0 \cap \Gamma\phi_0$ and some chamber ϕ in X_0 that is epicormic at $\mathcal{T}_n$. Let m be the length of α.

By Lemma 7 and Lemma 10 above, the gallery α projects to a gallery β in Y between ϕ_0 and a chamber ϕ_Y that is epicormic at $\mathcal{T}_n \cap Y$. The gallery β in Y has length at most m.

It follows from (2) of Lemma 9 above that the gallery β in Y has length greater than n. Therefore $m > n$. Hence the gallery-connected component of $D(n)$ that contains ϕ_0 is contained in X_0. As the chamber $\gamma \cdot \phi_0$ is not in X_0, it follows that the complex $D(n)$ is not connected. $\square$

This completes the proof, as Γ is finitely generated if and only if $D(n)$ is connected for some n.

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