Sydney University Mathematical Society Problem Competition 2010

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Entrants may use any source of information except other people. The problems will also be posted on the web page **http://www.maths.usyd.edu.au/u/SUMS/**.

Entrants may submit solutions to as many problems as they wish. Prizes ($60 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years). Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, preference may be given to students in the earlier year of university; otherwise, prizes may be shared. If a problem receives no correct solutions, its prize-money will be redistributed among the other problems.

Entries must be received by **Friday, August 13, 2010**. They may be posted to Dr Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or handed in to Room 805, Carslaw Building. Please mark your entry SUMS Problem Competition 2010, and include your name, university, student number, year of study, and postal address (or email address for University of Sydney students) for the return of your entry and prizes.

---

1. For any positive integer n, let $D(n)$ be the number obtained by writing next to each other the usual decimal expressions for $2n$ and for n, in that order. For example, $D(10) = 2010$ and $D(627) = 1254627$. Show that there are infinitely many n for which $D(n)$ is a perfect square.

2. Start with any nonempty string of (lowercase) letters. Apply the following operation: remove the first letter, and then after every other letter in the string, insert the letter which succeeds that letter in the alphabet, except that you should not insert anything after `z`. For example, the string `fsaazn` becomes `stababzno` after applying this operation, and that becomes `tuabbcabbcznoop` after applying the operation again. Show that, no matter what the initial string is, repeating this operation eventually results in the empty string.

3. Define $f(x)$ to be the sum of the series $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \cdots$, which converges for all real x. Show that for any positive integer m, the following equation holds for all real x:

$$f(x) + f(x + \frac{2\pi}{m}) + f(x + \frac{4\pi}{m}) + f(x + \frac{6\pi}{m}) + \cdots + f(x + \frac{2(m-1)\pi}{m}) = \frac{1}{m} f(mx).$$

4. The *Catalan numbers* are defined by the recursion $c_n = c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0$, with $c_0 = 1$. Determine the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{2^{4n}(2n+3)}$.

5. Fix an integer $n \geq 3$.

a) Construct a subset $S \subseteq \{1, 2, \cdots, n\}$ which is as large as possible such that among any three elements of S, there are two which have no common factor greater than 1.

b) Construct a subset $T \subseteq \{1, 2, \cdots, n\}$ which is as large as possible such that among any three elements of T, there are two which have a common factor greater than 1.

In each part, you must prove that no subset with more elements has the specified property.

6. The sisters Alice, Bess and Cath need to share a circular pizza which has been divided into $2n$ pieces (each a circular sector having an angle of $\frac{1}{n} \times 180^\circ$ at the centre of the pizza), where n is some integer greater than 1. An allocation of the $2n$ pieces to the three girls is acceptable if:

a) there is some diameter d of the pizza (that is, some line through the centre of the pizza) such that Alice's pieces all lie on the same side of d;

b) there is no diameter e of the pizza such that all the pieces on one side of e go to Bess;

c) every sister gets at least one piece.

Show that there are just as many acceptable allocations in which Cath gets an even number of pieces as there are in which she gets an odd number of pieces.

7. Find all real numbers x, y, z, t such that

$$x + y + z + t = x^2 + y^2 + z^2 + t^2 = x^3 + y^3 + z^3 + t^3 = x^4 + y^4 + z^4 + t^4.$$

8. Consider $n \times n$ matrices with entries in the field $\mathbb{Z}_p$ of integers modulo p, where p is some prime number. If M is such a matrix, its *characteristic polynomial* is defined to be $\det(xI - M)$ where I denotes the identity matrix, and we say that M is *unipotent* if its characteristic polynomial equals $(x-1)^n$ (that is, M has a single eigenvalue 1 with multiplicity n). Find, in terms of n and p, the smallest positive integer a such that $M^a = I$ for all unipotent matrices M.

9. A *ring* is a set R with two binary operations $(r, s) \mapsto r + s$ and $(r, s) \mapsto rs$, an operation $r \mapsto -r$, and an element $0 \in R$, satisfying the following axioms:

$$r + (s + t) = (r + s) + t, \quad r + s = s + r, \quad r + 0 = r, \quad r + (-r) = 0,$$
$$r(st) = (rs)t, \quad r(s + t) = rs + rt, \quad (r + s)t = rt + st,$$

for all $r, s, t \in R$. Show that if R is a ring with the property that $r^4 = r$ for all $r \in R$, then R is *commutative* in the sense that $rs = sr$ for all $r, s \in R$.

10. For any 2×2 integer matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the *integral column space* $C(A)$ of A to be

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; \begin{pmatrix} u \\ v \end{pmatrix} = s \begin{pmatrix} a \\ c \end{pmatrix} + t \begin{pmatrix} b \\ d \end{pmatrix} \text{ for some } s, t \in \mathbb{Z} \right\}.$$

a) Suppose that $A_1, A_2, \cdots, A_m$ are 2×2 integer matrices such that $\det(A_k) \neq 0$ for every k, and the union $C(A_1) \cup C(A_2) \cup \cdots \cup C(A_m)$ is all of $\mathbb{Z}^2$. Show that

$$\frac{1}{|\det(A_1)|} + \frac{1}{|\det(A_2)|} + \cdots + \frac{1}{|\det(A_m)|} \geq 1.$$

b) Show that for any $\epsilon > 0$, there is a sequence $A_1, A_2, A_3, \cdots$ of 2×2 integer matrices such that $\det(A_k) \neq 0$ for every k, the union $C(A_1) \cup C(A_2) \cup C(A_3) \cup \cdots$ is all of $\mathbb{Z}^2$, and

$$\frac{1}{|\det(A_1)|} + \frac{1}{|\det(A_2)|} + \frac{1}{|\det(A_3)|} + \cdots \leq \epsilon.$$