Sydney University Mathematical Society
Problems Competition 2002

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($50 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years). Entries from fourth year students will be considered. When prizewinners are being selected, if two or more entries to a problem are essentially equal, then preference may be given to the students in the earlier year of university.

Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, September 13, 2002. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **2002 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. Which is the larger of the two numbers

$$9^{9^{\cdot^{\cdot^{\cdot^{9}}}}} \quad (10\ 9\text{'s}), \quad \text{and} \quad 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \quad (9\ 10\text{'s})?$$

2. Consider three circles C_1, C_2 and C_3, with radii r_1, r_2 and r_3, respectively. Suppose that any two of them touch externally (i.e., have a common point, but disjoint interiors). Suppose that C_1 and C_3 have a common tangent line ℓ, and that C_2 and C_3 have a common tangent line ℓ' parallel to ℓ. Express r_3 in terms of r_1 and r_2.

3. Let H be the orthocentre of a triangle $\triangle ABC$, i.e., the intersection of the altitudes of the triangle. Now let X and Y be points on the sides BC and AC, respectively. Form the circles with diameters AX and BY, and let P and Q be their points of intersection. Show that P, Q and H are collinear.

4. "Two positive integers chosen at random are more likely to be relatively prime than not". True or false?

5. Describe the polynomials $p(X)$ with integer coefficients which have the property that $p(X^2) \equiv p(X)p(-X)$.

6. Suppose that we have n squares of total area at least 3, with their sides parallel to the x- and y-axes. Show that, moving them so that their sides remain parallel to the x- and y-axes, they can be made to cover the unit square $[0,1] \times [0,1]$.

7. Calculate

$$\int_0^1 \phi(x)\, dx \quad \text{and} \quad \int_0^1 x\phi(x)\, dx,$$

where $\phi(x)$ is any concave-down function defined on the interval $[0, \infty)$ satisfying the constraints, $\phi(0) = 0$ and $\phi(x+1) = \phi(x) + 1/(x+1)$ for all $x \geq 0$. The results involve Euler's constant γ defined by

$$\gamma = \lim_{n\to\infty} \big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\big).$$

8. This problem is about *polyhedra*, which may admit *symmetries* such as rotations and reflections. For example a cube admits a rotation of 90° about an axis through the centre of two opposite faces; it also admits a reflection in the horizontal plane mid-way between the top and the bottom faces.

We define a polyhedron to be *face-transitive* if, for any pair of faces, there is a symmetry which carries the first face onto the second. Similarly, a polyhedron is *vertex-transitive* if, for any pair of vertices, there is a symmetry which carries the first vertex onto the second; it is *edge-transitive* if, for any pair of edges, there is a symmetry which carries the first edge onto the second. (A cube is in fact face-transitive, vertex-transitive and edge-transitive.)

(i) Find a polyhedron that is face-transitive but not vertex-transitive.

(ii) Find a polyhedron that is vertex-transitive but not face-transitive.

(iii) Find a polyhedron that is both face-transitive and vertex-transitive but not edge-transitive.

(iv) Prove that an edge-transitive polyhedron must be either face-transitive or vertex-transitive.

9. Consider the rational number

$$x_n = \frac{2^1}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^n}{n}.$$

Write $x_n = 2^{v_n} a_n / b_n$, where a_n and b_n are odd integers (thus v_n is the "2-adic valuation" of x_n). Show that $v_n \to \infty$ as $n \to \infty$.

10. Consider the Van der Monde determinant

$$V = \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

in 3 variables x_1, x_2, x_3. Form all the partial derivatives

$$\frac{\partial^{i+j+k} V}{\partial^i x_1 \partial^j x_2 \partial^k x_3},$$

where $i, j, k \geq 0$ (which are all polynomials in the variables x_1, x_2, x_3). Show that we can find six of these, say $p_1, \ldots, p_6$, so that any of the other partial derivatives can be written as a linear combination $a_1 p_1 + \cdots + a_6 p_6$, for some constants $a_1, \ldots, a_6$. Generalize this statement to the case of the Van der Monde determinant in the n variables $x_1, \ldots, x_n$, showing that there are $n!$ partial derivatives so that all others are linear combinations of these ones.