Random Times and Enlargements of Filtrations

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

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April, 2012

To My Family

"Believe in me, because I was made for chasing dreams."

Staind - Illusion of Progress

Acknowledgement

"If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy."

- *Alfréd Rényi, 1921 – 1970*

Although the path of research is largely a lonesome one, it would not have been possible without the generous support of numerous individuals to whom I am intellectually and personally indebted.

However, before I take this opportunity to thank the many people who have made this thesis possible. I must first apologize to my grandparents to whom I bear the guilt of not been able to fulfill my duties as a grandson.

I must gratefully acknowledge my parents for providing me with the opportunity to pursue my selfish endeavours and to my mother for her quiet, everyday support that enabled me to focus on writing my thesis.

I must reserve the bulk of my gratitude for my friends Elise, Alister, Ken, Laurence, Jon, Jaq, Bao and Huiming whose friendship, empathy and understanding continues to transcend both time and geography. I must thank my friends in Sydney for keeping me alive and sane. Thank you Belina, Sheryl and Emi for feeding me when I am hungry, for looking after me when I am sick and for comforting me when I am down. Thank you Etienne, John and Jay for entertaining me when I am bored, for driving me when I am in need and for distracting me when I am working.

I must humbly thank all my teachers, both past and present, whose knowledge and enthusiasm helped illuminate these pages. Thank you in particular to Trevor Stanton for providing me with both mathematical stimulus and refuge in a foreign land. Thank you to Ian Doust and Benjamin Goldys for their immeasurable assistance and patience which considerably enriched my honours experience.

Thank you to Pavel V. Gapeev and Monique Jeanblanc for drafting the joint paper which manifested into Chapter 3 of this thesis. It is only with their insights that I was able to extend and complete the paper with the help of my supervisor Marek Rutkowski. Thank you Shiqi Song and again to Monique Jeanblanc for their comments and inputs which helped greatly the development of Chapter 6.

Finally, I am extremely grateful to Monique Jeanblanc and especially to my supervisor Marek Rutkowski for all their valuable time and inputs which helped to fill the pages of this thesis. It is only with their patience and guidance that I was able to complete my studies. Thank you both for helping me along this road of mathematical research, a road which I hope to continue for quite some times to come.

Contents

Preface

"C'est par la logique qu'on démontre, c'est par l'intuition qu'on invente." [1]

- *Henri Poincaré, 1854 – 1912*

This thesis is based on series of research papers co-authored by the candidate during his PhD studies. They include Gapeev et al. [44], Li and Rutkowski [76, 74, 75, 73] and Jeanblanc et al. [58]. Each chapter of this thesis is in fact self-contained with the main theme revolving around the study of random times, enlargements of filtration and construction of market models.

Historical background

The first part of this thesis is devoted to the study of enlargement of filtrations. The study of enlargements of filtrations was motivated by the following question: is the semimartingale property invariant with respect to a given enlargement $\mathbb{G}$ of the base filtration $\mathbb{F}$? In other words, is any $(\mathbb{P},\mathbb{F})$-semimartingale a $(\mathbb{P},\mathbb{G})$-semimartingale where $\mathbb{G}$ is a fixed filtration such that $\mathbb{F}\subset\mathbb{G}$? It can be shown that the answer to this question is negative, in general. If the answer is positive then we say that the hypothesis (H') holds between $\mathbb{F}$ and $\mathbb{G}$. If the hypothesis (H') holds, one is also interested in the semimartingale decomposition of an $\mathbb{F}$-semimartingale with respect the enlarged filtration $\mathbb{G}$.

Traditionally, the literature has focused on the *initial* and *progressive enlargement* of $\mathbb{F}$ with a single finite valued random time τ. These two types of enlargements has been studied extensively in the works including, but not limited to, Jeulin [62, 63, 64], Jeulin and Yor [65, 66], Jacod [50], Yor [107, 108] and Meyer [86]. In the study of initial enlargement, the standard assumption is the Jacod's criterion (see [50]), which assumes the conditional distribution of the random time τ is absolutely continuous with respect to a positive measure on $\mathbb{R}_+$. Within this framework, it can shown that the hypothesis (H') is indeed satisfied between $\mathbb{F}$ and $\mathbb{G}$ and the $\mathbb{G}$-semimartingale decomposition of $\mathbb{F}$-martingale can be derived explicitly.

On the other hand, the study of progressive enlargement has focused mainly on honest time with the main references been Jeulin [62, 63], Jeulin and Yor [66]. More recently, we have the works of Gasbarra et al. [45], Jeanblanc and Le Cam [57], Jeanblanc and Song [59, 60] and El Karoui et al. [36]. In the works of Gasbarra et al. [45], Jeanblanc and Le Cam [57] and El Karoui et al. [36], the authors studied the Jacod's criterion (also referred to as the *density hypothesis*) in the context of progressive enlargement and have derived the $\mathbb{G}$-compensator of $H=\mathbb{1}_{[\![\tau,\infty[\![}$ and the $\mathbb{G}$-semimartingale decomposition of $\mathbb{F}$-martingales. The works of Jeanblanc and Song [59, 60] answer similar questions that are investigated in Chapters 3, 4 and 7 of this thesis, but with different techniques and they aim to describe a larger class of models for conditional distributions, rather than focusing on *the hypothesis* (HP) (see Definition 4.2.6).

The second part of the thesis is devoted to the construction of Market Models for forward swap rates and Credit Default Swaps (CDS) spreads. The market model for forward LIBORs was first examined in papers by Brace et al. [19] and Musiela and Rutkowski [89]. Their approach was

[1] It is by logic that we prove, but by intuition that we discover.

subsequently extended by Jamshidian in [53, 54] to the market model for co-terminal forward swap rates. Since then, several papers on alternative market models for LIBORs and other families of forward swap rates were published.

To the best of our knowledge, there is relatively scarce financial literature in regard either to the existence or to methods of construction of market models for forward CDS spreads. This apparent gap is a bit surprising, especially when confronted with the market practitioners approach to credit default swaptions, which hinges on a suitable variant of the Black formula. As background readings, the reader are directed to the works of Galluccio et al. [42], Pietersz and Regenmortel [95], Rutkowski [98], or the monographs by Brace [18] or Musiela and Rutkowski [90] and the references therein.

Original contributions

We first outline the problems addressed in the thesis and the contributions. The purpose of preliminary Chapters 1 and 2 is to give an overview of the classic results in the general theory of processes and enlargements of filtrations. For an extensive study of the general theory, the reader is referred to, e.g., Dellacherie [30] or He et al. [48]. The reader familiar with the background material is thus advised to skip the first two chapters and move directly to Chapter 3.

Chapters 3 and 4, based on the papers Gapeev et al. [44] and Li and Rutkowski [74], focus on the following main question: given a supermartingale G, which is assumed to be non-negative and bounded by one, can we show that there exists a random time such that the *Azéma supermartingale* associated with this random time is given by G? In the context of random times and their applications to credit risk modeling, the above question was partially answered in a joint paper by Gapeev et al. [44] and, using a different method, in papers by Jeanblanc and Song in [59, 60] under the assumption that a supermartingale G is strictly positive and satisfies certain additional continuity assumptions. Their research was continued by Li and Rutkowski in [74], who answered the above-mentioned question in full generality with the help of the concept of a multiplicative system introduced by Meyer [85]. Sadly, when one took a closer look in the literature, it was mentioned by Meyer on page 186 of [82] that the original question was to some extent answered in a different context and using a different method by Blumenthal and Getoor [16]. With the above in view, the main contributions in Chapters 3 and 4 are:

- alternative constructions of a random time with a given in advanced Azéma supermartingale,
- the proof of the existence of multiple random times that are consistent with a given in advanced family of Azéma supermartingales,
- the proof of the uniqueness of conditional distribution under hypothesis (HP),
- a result showing that an honest time is an $\mathcal{F}_\infty$-measurable random time satisfying the hypothesis (HP).

In Chapters 5 and 6, we focus on the theory of enlargement of filtrations by placing ourselves in the usual filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the base filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We work specifically under the progressive enlargement of $\mathbb{F}$ with τ. The aim of Chapter 5, derived from Li and Rutkowski [75], is to study, in the context of the theory of progressive enlargement of filtration, the random times constructed in Chapter 3 and 4. Similarly as in the classic paper of Jeulin [62] and the recent paper by Jeanblanc and Song [60], our goal is to examine the semimartingale decomposition of $\mathbb{F}$-martingales in the progressive enlargement $\mathbb{G}$ of $\mathbb{F}$. Unfortunately, one must point out that we were unable to show that the hypothesis (H') is satisfied between $\mathbb{F}$ and the progressive enlargement $\mathbb{G}$ under only the hypothesis (HP). The contributions of this chapter to the current literature can be summarised as follows:

- the semimartingale result derived in this chapter extends the result of Jeanblanc and Song [59] by obtaining the form of the $\mathbb{G}$-semimartingale decomposition for a pseudo-honest time with a strictly positive conditional distribution, while removing continuity assumptions,

- we establish the $\mathbb{G}$-semimartingale decomposition in the case where τ is constructed using the predictable multiplicative system introduced in Meyer [85]; it appears that it resembles closely the classical $\mathbb{G}$-semimartingale decomposition for an honest time and, in fact, it reduces to the classical decomposition if we assume that either Property **(C)** or **(A)** holds (see Definition 1.5.4),

- inspired by Meyer [85], we introduce the optional multiplicative system and derive the $\mathbb{G}$-semimartingale decomposition when the random time is constructed through the optional multiplicative system,

- we show that the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{G}$ if τ satisfies the *extended density hypothesis* (see Definition 5.2.5) and derive the $\mathbb{G}$-semimartingale decomposition of $\mathbb{F}$-martingales.

In Chapter 6, we continue working under the setting of progressive enlargement and assume that one is given two random times both which satisfy either the hypothesis (H) or the hypothesis (H'). The question of interest is therefore under which conditions are the hypothesis (H) or the hypothesis (H') stable under taking minimum and maximum. More specifically, we follow some preliminary works done in [58] to show under which conditions is the hypothesis (H) or hypothesis (H') satisfied between $\mathbb{F}$ and the progressive enlargement of $\mathbb{F}$ with $\tau_1 \wedge \tau_2$ and/or $\tau_1 \vee \tau_2$. The main contributions of this chapter are the following results:

- a generalized version of the Norros lemma (see Proposition 3.1 in [43]),

- the proof of stability of the hypothesis (H) under minimum and maximum when the random times are *Cox-times* (see Definition 6.2.1),

- the proof of stability of the hypothesis (H') under minimum and maximum for arbitrary random times.

Diverging from the previous chapters, we focus in Chapters 7 and 8 on examining the necessary and sufficient conditions for the existence of market models for forward swap rates and CDS spreads which are applicable in the pricing of exotic instruments such as (credit default) swaptions. Given a family of forward swap rates, the aim of Chapter 7 is to first re-examine and extend the works of Galluccio et al. [42] and Pietersz and van Regenmortel [95]. Chapter 8 is in the same vein as Chapter 7, but we work under a defaultable framework and concentrate on extensions of CDS market models presented in Brigo [22, 23] and extend the results to a semimartingale framework. This part of the thesis contributes to the existing literature in several respects:

- we provide counter-examples to some results of Galluccio et al. [42],

- the results of Pietersz and van Regenmortel [95] on the positivity of bonds within a class of market model are clarified,

- necessary and sufficient conditions for the existence of a market model consistent with the given in advanced family of forward swaps and derive are given,

- the joint dynamics of forward swap rates under a single probability measure are derived,

- a systematic study of the market models for CDS spreads is presented and the joint dynamics of a family of CDS spreads are derived in each case.

To conclude, one must point out that the derivation of the joint dynamics of forward swap rate or the CDS spreads does not justify by itself that the model is suitable for the arbitrage pricing of credit derivatives. To show the viability of the model, it would be sufficient to demonstrate that the model arbitrage-free in some sense, for instance, that it can be supported by an associated arbitrage-free model for default-free and defaultable zero-coupon bonds.

Overview of chapters

- **Chapter 1.** We introduce, for the purpose of this thesis, the most pertinent results from the general theory of stochastic processes.

- **Chapter 2.** We give a brief overview of the theory of enlargements of filtrations. The basics of initial and progressive enlargement of a filtration with a random time are introduced in Sections 2.2 and 2.3, respectively, while some classic results regarding the semimartingale decomposition under the progressive enlargement with a random time or an honest time are presented in Section 2.5. An original result is Proposition 2.2.1, which extends Jacod's criterion to the case where the $\mathbb{F}$-conditional distribution satisfies the random Lipschitz condition.

- **Chapter 3.** We follow here the work of Gapeev et al. [44] to provide an explicit construction of a random time when the associated Azéma semimartingale is given in advance. Our approach hinges on the use of a variant of Girsanov's theorem combined with a judicious choice of the Radon-Nikodým density process. The proposed solution is also partially motivated by the classic example arising in the filtering theory.

- **Chapter 4.** We present the work by Li and Rutkowski [74], which was motivated by Gapeev et al. [44] and the recent results of Jeanblanc and Song [59, 60]. Our aim is to demonstrate, with the help of multiplicative systems introduced in Meyer [85], that for any given positive $\mathbb{F}$-submartingale F such that $F_\infty = 1$, there exists a random time τ on some extension of the filtered probability space such that the Azéma submartingale associated with τ coincides with F. Pertinent properties of this construction are studied and it is subsequently extended to the case of several correlated random times with the predetermined univariate conditional distributions and an arbitrary correlation structure given by a choice of a copula function.

- **Chapter 5.** This chapter derives from Li and Rutkowski [75], where we deal with various alternative decompositions of $\mathbb{F}$-martingales with respect to the filtration $\mathbb{G}$ which represents the enlargement of a filtration $\mathbb{F}$ by a progressive flow of observations of a random time that either belongs to the class of pseudo-honest times or satisfies the extended density hypothesis. Several related results from the existing literature are essentially extended. We outline two potential applications of our results to specific problems arising in Financial Mathematics.

- **Chapter 6.** We move away from a single random time and we present some preliminary results obtained by Jeanblanc, Li and Song [58]. The motivation for this chapter derives from the well-known fact that the minimum and maximum of two stopping times is again a stopping time in the same filtration. Therefore, working under progressive enlargement, the aim is to provide sufficient and/or necessary conditions for the stability of the hypothesis (H) and (H') under minimum and maximum of two random times.

- **Chapter 7.** Following Li and Rutkowski [76], we re-examine and extend certain results from the papers by Galluccio et al. [42] and Pietersz and van Regenmortel [95]. We establish several results providing alternative necessary and sufficient conditions for admissibility of a family of forward swaps, that is, the property that it is supported by a family of bonds associated with the underlying tenor structure. We also derive the generic expression for the joint dynamics of a family of forward swap rates under a single probability measure and we show that these dynamics are uniquely determined by a selection of volatility processes with respect to the set of driving martingales.

- **Chapter 8.** Following Li and Rutkowski [73], we provide the construction of several variants of market models for forward CDS spreads, as first presented by Brigo [23]. We compute explicitly the joint dynamics for some families of forward CDS spreads under a common probability measure. We first examine this problem for single-period CDS spreads under certain simplifying assumptions. Subsequently, we derive, without any restrictions, the joint dynamics under a common probability measure for the family of one- and two-period forward CDS spreads, and the family of one-period and co-terminal forward CDS spreads.

Chapter 1

Elements from the General Theory of Stochastic Processes

The purpose of this chapter is to introduce the most pertinent results from the general theory of stochastic process for the purpose of this thesis. Most results are presented here without proofs and they can be found in Dellacherie [30] and He et al. [48]. It is assumed in this thesis that the reader is familiar with the theory of martingales and stochastic integration.

1.1 Stochastic Processes and Stopping Times

This section is devoted to basic properties of stopping times and related results for stochastic processes. We assume throughout that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F}$, satisfying the usual conditions.

Definition 1.1.1. A *random time* τ is a map from Ω to $\overline{\mathbb{R}}_+$. A random time is a $\mathbb{F}$-*stopping time* if $\{\tau \leq t\}$ is an element of $\mathcal{F}_t$ for all $t \geq 0$.

Definition 1.1.2. A random time τ is said to *avoid* all $\mathbb{F}$-stopping times, if for all $\mathbb{F}$-stopping time T, we have $\mathbb{P}(\tau = T) = 0$.

Definition 1.1.3. A stopping time T is predictable, if there exist a monotone increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} T_n = T$$

Definition 1.1.4. Given a $\mathbb{F}$-stopping time T, the σ-algebras $\mathcal{F}_T$ and $\mathcal{F}_{T-}$ are defined as follows

$$\mathcal{F}_T = \{A \in \mathcal{F} \,|\, A \cap \{T \leq t\} \in \mathcal{F}_t\,,\, \forall t \geq 0\}$$
$$\mathcal{F}_{T-} = \mathcal{F}_0 \vee \sigma\{A \cap \{T > t\} \text{ where } A \in \mathcal{F}_t \text{ and } t \geq 0\}$$

Proposition 1.1.1. *Let T be a stopping time. Then $T \in \mathcal{F}_{T-} \subset \mathcal{F}_T$.*

Proposition 1.1.2. *If S and T are two stopping times then*

1. *For all $A \in \mathcal{F}_S$, the set $A \cap \{S \leq T\} \in \mathcal{F}_T$*
2. *For all $A \in \mathcal{F}_S$, the set $A \cap \{S < T\} \in \mathcal{F}_T$*

In particular, if $A = \Omega$ then $\{S \leq T\}$ belongs to $\mathcal{F}_T$ and $\{S < T\}$ belongs to $\mathcal{F}_{T-}$

Proposition 1.1.3. *If S and T are two stopping times such that $S \leq T$ then $\mathcal{F}_S$ $(\mathcal{F}_{S-})$ is a sub σ-algebra of $\mathcal{F}_T$ $(\mathcal{F}_{T-})$*

Theorem 1.1.1. *Let $(T_n)_{n\in\mathbb{N}}$ be a monotone sequence of stopping times such that $T = \lim_n T_n$.*
(i) If $T_n \downarrow T$ then

$$\mathcal{F}_T = \bigwedge_n \mathcal{F}_{T_n}.$$

(ii) If $T_n \uparrow T$ then

$$\mathcal{F}_{T^-} = \bigvee_n \mathcal{F}_{T_n -}$$

Corollary 1.1.1. *Let $(T_n)_{n\in\mathbb{N}}$ be a monotone sequence of stopping times such that $\lim_n T_n = T$.*
(i) If a sequence $(T_n)_{n\in\mathbb{N}}$ is decreasing on the set $\{0 < T_n < \infty\}$ for all n then

$$\mathcal{F}_T = \bigwedge_n \mathcal{F}_{T_n -}$$

(ii) if a sequence $(T_n)_{n\in\mathbb{N}}$ is increasing on the set $\{0 < T_n < \infty\}$ for all n then

$$\mathcal{F}_{T-} = \bigvee_n \mathcal{F}_{T_n}$$

Proof. The result is an application of Theorem 1.1.1 and Proposition 1.1.2 □

The above theorems generalizes the definition of $\mathcal{F}_{t-}$, where $t \in \mathbb{R}_+$.

Theorem 1.1.2. *(Optional Stopping Theorem) Let X be a right-continuous and uniformly integrable martingale, then for any stopping time S and T such that $S \leq T$,*

$$\mathbb{E}_{\mathbb{P}}\left(X_T \,\middle|\, \mathcal{F}_S\right) = X_S$$

Theorem 1.1.3. *(Predictable Stopping Theorem) Let X be a right-continuous and uniformly integrable martingale, then for any predictable stopping times S and T such that $S \leq T$,*

$$\mathbb{E}_{\mathbb{P}}\left(X_T \,\middle|\, \mathcal{F}_{T-}\right) = \mathbb{E}_{\mathbb{P}}\left(X_\infty \,\middle|\, \mathcal{F}_{T-}\right) = X_{T-}.$$

Corollary 1.1.2. *If the filtration $(\mathcal{F}_t)_{t\geq 0}$ is continuous then all $\mathbb{F}$-martingales are continuous.*

Proof. Note that $T = t$ is a predictable stopping time. Hence by Theorem 1.1.3 and continuity of filtration ($\mathcal{F}_t = \mathcal{F}_{t-} = \bigvee_n \mathcal{F}_{t_n}$).

$$\begin{aligned} X_{t-} &= \mathbb{E}_{\mathbb{P}}\left(X_t \,\middle|\, \mathcal{F}_{t-}\right) \\ &= \mathbb{E}_{\mathbb{P}}\left(X_t \,\middle|\, \mathcal{F}_t\right) \\ &= X_t \end{aligned}$$

Therefore X is continuous. □

Theorem 1.1.4. *(Doob-Meyer decomposition) Let X be a submartingale of class (D). Then there exists unique processes M and A such that*

$$X = X_0 + M + A$$

where M is a local martingale and A is a predictable process of finite variation.

Remark 1.1.1. One adopted in Theorem 1.1.4 the convention that $M_0 = 0$ and $A_0 = 0$.

1.2 Predictable and Optional Sets

In this section, we work on the space $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), \mathbb{P} \times \lambda)$ where λ is the Lebesgue measure on $\mathbb{R}_+$. The aim is to introduce the notions of the *predictable* and *optional* σ-algebras on $\Omega \times \mathbb{R}_+$.

Definition 1.2.1. Given a filtration $\mathbb{F}$, we define the following σ-algebras on $\Omega \times \mathbb{R}_+$
(i) the σ-algebra $\mathcal{O}(\mathbb{F})$, which is generated by all $\mathbb{F}$-adapted càdlàg processes.
(ii) the σ-algebra $\mathcal{P}(\mathbb{F})$. which is generated by all $\mathbb{F}$-adapted càglàd processes.
The σ-algebra $\mathcal{O}(\mathbb{F})$ is the $\mathbb{F}$-*optional σ-algebra* and $\mathcal{P}(\mathbb{F})$ is the $\mathbb{F}$-*predictable σ-algebra*.

For brevity, we shall often write $\mathcal{O}$ for $\mathcal{O}(\mathbb{F})$ ($\mathcal{P}$ for $\mathcal{P}(\mathbb{F})$), if there is no confusion with the filtration which we work on.

Lemma 1.2.1. *The predictable σ-algebra is contained inside the optional σ-algebra.*

Proof. Suppose that X is càglàd and $\mathbb{F}$-adapted, so that $X \in \mathcal{P}$. We define

$$X_t^{(n)} = \sum_{k=0}^{\infty} X_{\frac{k}{2^n}} \mathbb{1}_{[\frac{k}{2^n}, \frac{k+1}{2^n})}(t)$$

Then $X_t^{(n)} \to X_t$ pointwise and thus X is measurable with respect to $\mathcal{O}$ because for each $n \in \mathbb{N}$ the process $X^{(n)}$ is measurable with respect $\mathcal{O}$. This shows that $\mathcal{P} \subset \mathcal{O}$. □

Definition 1.2.2. Let T and S be two stopping times, such that $S \le T$. The set $[\![S, T [\![$ included in $\Omega \times \mathbb{R}_+$ and given by the following is called a *stochastic interval*

$$[\![S, T [\![= \{(\omega, t) \,|\, S(\omega) \le t < T(\omega)\}$$

The stochastic intervals $[\![S, T]\!]$, $]\!] S, T [\![$ and $]\!] S, T]\!]$ are defined similarly. The stochastic interval $[\![T]\!]$ is also called the *graph* of T.

Lemma 1.2.2. *(i) The predictable σ-algebra can also be characterised by the following*

$$\mathcal{P}(\mathbb{F}) = \{A \times \{0\} \,:\, A \in \mathcal{F}_0\} \vee \{A \times [s,t) \,:\, 0 < s < t,\ s,t \in \mathbb{Q}_+,\ A \in \bigvee_{r<s} \mathcal{F}_r\}.$$

(ii) The optional σ-algebra can also be characterised by the following equality

$$\mathcal{O} = \sigma\{[\![S, \infty [\![\,|\, S \text{ is a stopping time}\}$$

Theorem 1.2.1. *(Optional Section Theorem) Let X and Y be two optional (predictable) processes. If for every (predictable) stopping time T, the random variables $X_T \mathbb{1}_{\{T<\infty\}}$ and $Y_T \mathbb{1}_{\{T<\infty\}}$ are integrable and the equality*

$$\mathbb{E}_{\mathbb{P}}\left(X_T \mathbb{1}_{\{T<\infty\}}\right) = \mathbb{E}_{\mathbb{P}}\left(Y_T \mathbb{1}_{\{T<\infty\}}\right)$$

holds then $X = Y$ up to an evanescent set.

1.3 Projection Theorems

In this section, we introduce the notion of the *optional* and *predictable projection* of a stochastic process.

Theorem 1.3.1. *Let X be a measurable process, such that for all stopping (predictable) time T, $X_T \mathbb{1}_{\{T<\infty\}}$ is integrable. Then*
(i) There exists a unique optional process ${}^{o}X$ such that, for all stopping times T,

$$\mathbb{E}_{\mathbb{P}}\left(X_T \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right) = {}^{o}X_T \mathbb{1}_{\{T<\infty\}}.$$

The process ${}^oX$ is called the $\mathbb{F}$-optional projection of X.
(ii) There exists a unique predictable process ${}^pX$ such that, for all predictable stopping times T,

$$\mathbb{E}_{\mathbb{P}}\left(X_T \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_{T-}\right) = {}^pX_T \mathbb{1}_{\{T<\infty\}}.$$

The process ${}^pX$ is called the $\mathbb{F}$-predictable projection of X.

Proposition 1.3.1. *Let X be a measurable process and Y an optional (predictable, resp.) process. If the optional (predictable, resp.) projection of X exists, then the optional (predictable, resp.) projection of XY exists and if given by*

$${}^o(XY) = ({}^oX)Y, \quad {}^p(XY) = ({}^pX)Y.$$

Proof. For the optional case, invoking the property of conditional expectation, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(X_T Y_T \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right) &= Y_T \mathbb{1}_{\{T<\infty\}} \mathbb{E}_{\mathbb{P}}\left(X_T \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right) \\ &= \mathbb{1}_{\{T<\infty\}} Y_T \, {}^oX_T.\end{aligned}$$

The proof for the predictable case is similar. □

Proposition 1.3.2. *Suppose that X is a measurable process. If the optional and predictable projections of X exist then ${}^p({}^oX) = {}^pX$.*

Proof. The proof follows by direct computations:

$$\begin{aligned}{}^p({}^oX_T)\mathbb{1}_{\{T<\infty\}} &= \mathbb{E}_{\mathbb{P}}\left({}^oX_T \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_{T^-}\right) \\ &= \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(X_T \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_T\right) \,\middle|\, \mathcal{F}_{T^-}\right) \\ &= \mathbb{E}_{\mathbb{P}}\left(X_T \mathbb{1}_{\{T<\infty\}} \,\middle|\, \mathcal{F}_{T^-}\right) \\ &= {}^pX \mathbb{1}_{\{T<\infty\}}\end{aligned}$$

where the third equality holds by Theorem 1.1.3. □

We conclude this section by providing an interesting characterisation of the optional and predictable σ-algebras.

Theorem 1.3.2. *Let $\mathcal{I}(\mathbb{F})$ be the σ-algebra generated by the jump processes ΔM_t where M ranges through all bounded $\mathbb{F}$-martingales. Then*

$$\mathcal{O}(\mathbb{F}) = \mathcal{P}(\mathbb{F}) \vee \mathcal{I}(\mathbb{F})$$

In particular, if all $\mathbb{F}$-martingales are continuous then $\mathcal{O}(\mathbb{F}) = \mathcal{P}(\mathbb{F})$ and every stopping time is predictable.

1.4 Dual Projections and Increasing Processes

In this section, we work on either the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\mathbb{F}$ satisfying the usual conditions or the product space $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), \mathbb{P} \times \lambda)$.

Let A be an (non-adapted) increasing process. We define a non-negative measure μ_A on the σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ by setting

$$\mu_A(H) := \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} \mathbb{1}_H(\omega, s)\, dA_s(\omega)\right), \quad H \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+).$$

We say that μ_A is the *measure generated by* the process A. The corresponding measure on the space of non-negative, bounded processes is defined by

$$\mu_A(X) := \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} X_s \, dA_s\right)$$

where X is any non-negative, bounded process.

Definition 1.4.1. A measure μ on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ is said to be an *optional measure* if for any non-negative, bounded process X

$$\mu(X) := \mathbb{E}_\mu(X) = \mathbb{E}_\mu\left({}^oX\right) =: \mu({}^oX)$$

where ${}^oX$ is the optional projection of X. A measure μ is said to be a *predictable measure* if for any non-negative, bounded process X

$$\mu(X) := \mathbb{E}_\mu(X) = \mathbb{E}_\mu\left({}^pX\right) =: \mu({}^pX)$$

where ${}^pX$ is the predictable projection of X.

Theorem 1.4.1. *Let A be a (non-adapted) increasing process. Then the measure μ_A generated by A on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ is an optional (predictable, resp.) measure if and only if A is an adapted (predictable, resp.) process.*

Definition 1.4.2. If μ is a σ-finite measure on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ then we define the measures μ^o and μ^p on the space of non-negative processes by setting

$$\mu^o(X) := \mu({}^oX), \qquad \mu^p(X) := \mu({}^pX)$$

Then μ^o (μ^p, resp.) is called the *optional* (*predictable*, resp.) *projection* of μ. It is clear from definition that μ^o (μ^p, resp.) is an optional (predictable, resp.) measure.

Remark 1.4.1. The restriction of μ to the optional σ-algebra $\mathcal{O}$ is equal to μ^o while the restriction of μ to $\mathcal{P}$ is equal to μ^p. A measure ν on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ is optional (predictable, resp.) if and only if $\nu = \nu^o$ ($\nu = \nu^p$, resp.).

Theorem 1.4.2. *Let μ_A be a measure generated by an increasing process A on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$. Then if A is a locally integrable process.*
(i) the optional measure μ_A^o is generated by a unique adapted increasing process,
(ii) the predictable measure μ_A^p is generated by a unique predictable increasing process,

Lemma 1.4.1. *(i) If A is a locally integrable increasing process then there exists a unique increasing optional process A^o such that, for any non-negative bounded process X,*

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} {}^oX_s \, dA_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} X_s \, dA_s^o\right).$$

The process A^o is called the dual predictable projection of A.
(ii) If A is a locally integrable increasing process then there exists a unique increasing predictable process A^p such that, for any non-negative bounded process X,

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} {}^pX_s \, dA_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} X_s \, dA_s^p\right).$$

The process A^p is called the dual predictable projection of A.

Proof. We only present the predictable case, since the proof for the optional case is similar. Let μ_A be the measure generated by A and μ_A^p be the predictable projection of μ_A. Theorem 1.4.2 gives the existence of a unique predictable, increasing process, which generates μ_A^p; we denote this process by A^p. Then, by definition,

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} {}^pX_s\, dA_s\right) = \mu_A({}^pX) = \mu_A^p(X) = \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty)} X_s\, dA_s^p\right).$$

□

Lemma 1.4.2. *(i) If A is a locally integrable, increasing process and S and T are two stopping times such that $S \leq T$ then the following holds*

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[S,T)} {}^oX_s\, dA_s \,\middle|\, \mathcal{F}_S\right) = \mathbb{E}_{\mathbb{P}}\left(\int_{[S,T)} X_s\, dA_s^o \,\middle|\, \mathcal{F}_S\right).$$

(ii) If A is a locally integrable, increasing process then the following holds

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[S,T)} {}^pX_s\, dA_s \,\middle|\, \mathcal{F}_{S-}\right) = \mathbb{E}_{\mathbb{P}}\left(\int_{[S,T)} X_s\, dA_s^p \,\middle|\, \mathcal{F}_{S-}\right).$$

Lemma 1.4.3. *Let A be a (non-adapted) process of integrable variation. Then the processes*

$${}^oA - A^o \quad \text{and} \quad {}^oA - A^p$$

are uniformly integrable martingales.

The following results shows how to calculate the jumps of the optional (predictable) dual projections.

Proposition 1.4.1. *(i) Let S be any stopping time. Then*

$$\Delta A_S^o \mathbb{1}_{\{S<\infty\}} = \mathbb{E}_{\mathbb{P}}\left[\Delta A_S \mathbb{1}_{\{S<\infty\}} \,|\mathcal{F}_S\right]$$

(ii) Let S be any predictable stopping time. Then

$$\Delta A_S^p \mathbb{1}_{\{S<\infty\}} = \mathbb{E}_{\mathbb{P}}\left[\Delta A_S \mathbb{1}_{\{S<\infty\}} \,|\mathcal{F}_{S-}\right]$$

1.5 Random Times and Related Processes

We will now apply the general theory of stochastic process to the study of finite random times, which are given on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with an arbitrary filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Our goal is also to introduce here the notation for several pertinent characteristics of a random time τ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Given a finite random time τ and the non-adapted process $H := \mathbb{1}_{[\![\tau,\infty[\![}$, we focus on the dual optional (predictable) projections of the process H. It is evident that the optional and predictable projection depends on the choice of filtration. Thus throughout the rest of this thesis, if there is a need to distinguish between projections on different filtrations, then we will adapt the following notation for the dual $\mathbb{F}$-optional projection and the dual $\mathbb{F}$-predictable projection of $H = \mathbb{1}_{[\![\tau,\infty[\![}$:

$$\bar{A}_t^{\tau,\mathbb{F}} := (H)_t^{o,\mathbb{F}}, \qquad A_t^{\tau,\mathbb{F}} := (H)_t^{p,\mathbb{F}}, \qquad t \in [0,\infty).$$

Furthermore, the $\mathbb{F}$-optional projections are denoted by:

$$F_t^{\tau,\mathbb{F}} := {}^{o,\mathbb{F}}(H)_t, \qquad G_t^{\tau,\mathbb{F}} := {}^{o,\mathbb{F}}(1-H)_t, \qquad t \in [0,\infty).$$

Since the random time τ is finite, one can show that $G_{\infty-}^{\tau,\mathbb{F}} = 0$ almost surely, and thus $F_{\infty-}^{\tau,\mathbb{F}} = F_{\infty}^{\tau,\mathbb{F}} = 1$.

Definition 1.5.1. Given a filtration $\mathbb{F}$, the process $G^{\tau,\mathbb{F}}$ is termed the *Azéma supermartingale* associated with the random time τ .

Proposition 1.5.1. *The process $G^{\tau,\mathbb{F}}$ is generated by $\bar{A}^{\tau,\mathbb{F}}$ or $A^{\tau,\mathbb{F}}$, that is*

$$G_t^{\tau,\mathbb{F}} = \mathbb{E}_{\mathbb{P}}\left(\bar{A}_\infty^{\tau,\mathbb{F}} - \bar{A}_t^{\tau,\mathbb{F}} \,\middle|\, \mathcal{F}_t\right) \quad \text{and} \quad G_t^{\tau,\mathbb{F}} = \mathbb{E}_{\mathbb{P}}\left(A_\infty^{\tau,\mathbb{F}} - A_t^{\tau,\mathbb{F}} \,\middle|\, \mathcal{F}_t\right) \tag{1.1}$$

for all $t \in [0,\infty)$.

Proof. Using Lemma 1.4.3, we can define the following $\mathbb{F}$-martingale associated with the dual $\mathbb{F}$-optional projection

$$\bar{m}_t^{\tau,\mathbb{F}} := F_t^{\tau,\mathbb{F}} - \bar{A}_t^{\tau,\mathbb{F}}, \qquad t \in [0,\infty).$$

Similarly, the $\mathbb{F}$-martingale associated with the dual $\mathbb{F}$-predictable projection is given by

$$m_t^{\tau,\mathbb{F}} := F_t^{\tau,\mathbb{F}} - A_t^{\tau,\mathbb{F}}, \qquad t \in [0,\infty).$$

It should be stressed that $\bar{m}_0^{\tau,\mathbb{F}}$ $(m_0^{\tau,\mathbb{F}})$ and $\bar{A}_0^{\tau,\mathbb{F}}$ $(A_0^{\tau,\mathbb{F}})$ are not necessarily zero and $\bar{m}^{\tau,\mathbb{F}}$ and $m^{\tau,\mathbb{F}}$ are both continuous at infinity. Since $F_\infty^{\tau,\mathbb{F}} = 0$, the processes $\bar{m}_t^{\tau,\mathbb{F}}$ and $m_t^{\tau,\mathbb{F}}$ take the form, $\bar{m}_t^{\tau,\mathbb{F}} = \mathbb{E}_{\mathbb{P}}\left(1 - \bar{A}_\infty^{\tau,\mathbb{F}} \,\middle|\, \mathcal{K}_t\right)$ and $m_t^{\tau,\mathbb{F}} = \mathbb{E}_{\mathbb{P}}\left(1 - A_\infty^{\tau,\mathbb{F}} \,\middle|\, \mathcal{K}_t\right)$ for $t \in [0,\infty)$. It is now easy to obtain the representation stated in (1.1). □

Definition 1.5.2. From Proposition 1.5.1, we obtain on $[0,\infty)$ the following decompositions:
(i) the *optional additive decomposition* of the Azéma supermartingale associated with τ

$$G_t^{\tau,\mathbb{F}} = \mathbb{E}_{\mathbb{P}}\left(\bar{A}_\infty^{\tau,\mathbb{F}} \,\middle|\, \mathcal{K}_t\right) - \bar{A}_t^{\tau,\mathbb{F}} = \bar{M}_t^{\tau,\mathbb{F}} - \bar{A}_t^{\tau,\mathbb{F}} \tag{1.2}$$

where $\bar{M}_t^{\tau,\mathbb{F}} := \mathbb{E}_{\mathbb{P}}\left(\bar{A}_\infty^{\tau,\mathbb{F}} \,\middle|\, \mathcal{K}_t\right)$.
(ii) the *predictable additive decomposition* of the Azéma supermartingale associated with τ

$$G_t^{\tau,\mathbb{F}} = \mathbb{E}_{\mathbb{P}}\left(A_\infty^{\tau,\mathbb{F}} \,\middle|\, \mathcal{K}_t\right) - A_t^{\tau,\mathbb{F}} = M_t^{\tau,\mathbb{F}} - A_t^{\tau,\mathbb{F}} \tag{1.3}$$

where $M_t^{\tau,\mathbb{F}} := \mathbb{E}_{\mathbb{P}}\left(A_\infty^{\tau,\mathbb{F}} \,\middle|\, \mathcal{K}_t\right)$.

Another process of interest is $V^{\tau,\mathbb{F}} := {}^{o,\mathbb{F}}(1 - \mathbb{1}_{]\!]\tau,\infty[\![})$. It is known from [62] that

$$V^{\tau,\mathbb{F}} = G^{\tau,\mathbb{F}} + \Delta\widehat{A}^{\tau,\mathbb{F}}, \qquad V_-^{\tau,\mathbb{F}} = G_-^{\tau,\mathbb{F}}, \qquad V_+^{\tau,\mathbb{F}} = G^{\tau,\mathbb{F}}. \tag{1.4}$$

The above equalities will later be useful in obtaining an optional multiplicative system associated with a random time τ.

For ease of presentation, whenever there is no danger of confusion with the random time τ and the filtration $\mathbb{F}$, we shall omit the superscripts ${}^{\tau,\mathbb{F}}$ on the (dual) projections of $H = \mathbb{1}_{[\![\tau,\infty[\![}$. We thus denote the dual optional and predictable projections of H by $\bar{A}$ and A, respectively. The optional projection of H will be denoted by F and the optional projection of $\mathbb{1}_{[\![0,\tau]\!]}$ by V. Therefore, equations (1.2) and (1.3) can be rewritten as

$$G_t = \bar{M}_t - \bar{A}_t,$$

for the optional additive decomposition and

$$G_t = M_t - A_t,$$

for the predictable additive decomposition of G. Consequently, the optional additive decomposition of F is given by

$$F_t = 1 - \bar{M}_t + \bar{A}_t,$$

and the predictable additive decomposition becomes

$$F_t = 1 - M_t + A_t,$$

Another important process under consideration will be the increasing process D, such that $0 \leq D_t \leq 1$ and D_t is $\mathcal{F}_\infty$-measurable. It is defined by setting, for all $t \in \bar{\mathbb{R}}_+$,

$$D_t := \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_\infty) = \mathbb{E}_{\mathbb{P}}(H_t \,|\, \mathcal{F}_\infty). \tag{1.5}$$

Note that we can recover the $(\mathbb{P}, \mathbb{F})$-submartingale F associated with τ since

$$F_t := \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t)$$

Recall that the supermartingale $G_t = 1 - F_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$ is called the Azéma supermartingale of τ. Therefore, we find it natural to refer to the $(\mathbb{P}, \mathbb{F})$-submartingale F as the *Azéma submartingale* of τ.

Definition 1.5.3. The $(\mathbb{P}, \mathbb{F})$-*conditional distribution of* τ is the random field $(F_{u,t})_{u,t \in \bar{\mathbb{R}}_+}$ given by

$$F_{u,t} := \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t), \quad \forall\, u, t \in \bar{\mathbb{R}}_+.$$

The $(\mathbb{P}, \mathbb{F})$-*conditional survival distribution of* τ is the random field $(G_{u,t})_{u,t \in \bar{\mathbb{R}}_+}$ given by

$$G_{u,t} := \mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = 1 - F_{u,t}, \quad \forall\, u, t \in \bar{\mathbb{R}}_+.$$

Note that the following equalities are valid, for all $t \leq u$,

$$F_{u,t} = \mathbb{E}_{\mathbb{P}}(H_u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F_u \,|\, \mathcal{F}_t) \tag{1.6}$$

and thus, in particular, the equality $F_t = F_{t,t}$ holds for all $t \in \bar{\mathbb{R}}_+$. It is also worth noting that $D_t = F_{t,\infty}$ for all $t \in \bar{\mathbb{R}}_+$. It is obvious that the random field $(F_{u,t})_{u,t \in \bar{\mathbb{R}}_+}$ provides more information about the probabilistic properties of a random time than the Azéma submartingale F. We will later argue that the knowledge of a process D also conveys more information about a random time than the Azéma submartingale F. This is intuitively clear, since when a random time τ is not known then F can always be recovered from D but, in general, the converse implication does not hold. It appears that, for a given Azéma submartingale F, one may find several increasing processes D such that D_t is $\mathcal{F}_\infty$-measurable and D *generates* F, meaning that the equality $F_t = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t)$ holds for all t.

In the following, we will sometimes refer to the following two assumptions, which are also frequently used in the existing literature.

Definition 1.5.4. (i) We say that a random time satisfies assumption (**A**) whenever $\mathbb{P}(\tau = S) = 0$ for all $\mathbb{F}$ stopping times S, meaning that τ avoids all $\mathbb{F}$ stopping times.
(ii) We say that assumption (**C**) is satisfied whenever all $(\mathbb{P}, \mathbb{F})$-local martingales are continuous.

Let us conclude this section with an application of Proposition 1.4.1. The following result provides a sufficient condition for the continuity of the dual optional (predictable) projection of H.

Lemma 1.5.1. *If a random time τ avoids all $\mathbb{F}$-stopping times then the $\mathbb{F}$-dual optional projection of $H = \mathbb{1}_{[\![\tau, \infty [\![}$ is continuous.*

Proof. By setting $A_t = H_t = \mathbb{1}_{\{\tau \leq t\}}$ in Proposition 1.4.1, we obtain

$$\mathbb{E}_{\mathbb{P}}\left(\Delta H^o_S \mathbb{1}_{\{S<\infty\}}\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau = S\}} \mathbb{1}_{\{S<\infty\}}\right) \leq \mathbb{P}(\tau = S) = 0,$$

since, by assumption, $\mathbb{P}(\tau = S) = 0$. ☐

Chapter 2

Enlargement of Filtration

The theory of enlargement of filtration has been studied in the past thirty years and is centered around the two following hypotheses:
(i) *The hypothesis* (H): any martingale in the smaller filtration is again a martingale in the larger filtration.
(ii) *The hypothesis* (H'): any semimartingale in the smaller filtration is again a semimartinagale in the larger filtration.

More specifically, the study of enlargement of filtration usually assumes the existence of a finite random time τ on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. It is then typical to study the hypothesis (H) and the hypothesis (H') under the assumption that the larger filtration is obtained by either the initial or the progressive enlargement with the random time τ (see Definition 2.2.1 and Definition 2.3.1).

The purpose of this chapter is to present several known results from the literature of enlargement of filtrations. Note that the main focus of this thesis is on the progressive enlargement with a random time, therefore, we shall only present only a few results on the initial enlargement, while the rest of the chapter is devoted to results on characterization of progressive enlargement and semimartingale decompositions. The main source of reference for this chapter are the works by Jacod [50], Jeulin [62, 63], Jeulin and Yor [65, 66] and Jeanblanc and Le Cam [57].

2.1 Preliminaries

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and we assume that τ is a random time with values in $\mathbb{R}_+$ defined on this space.

Definition 2.1.1. By an *enlargement of* $\mathbb{F}$ *associated with* τ (or, briefly, an *enlargement of* $\mathbb{F}$) we mean any filtration $\mathbb{K} = (\mathcal{K}_t)_{t\in\mathbb{R}_+}$ in $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the usual conditions, and such that: (i) the inclusion $\mathbb{F} \subset \mathbb{K}$ holds, meaning that $\mathcal{F}_t \subset \mathcal{K}_t$ for all $t \in \mathbb{R}_+$, and (ii) τ is a $\mathbb{K}$-stopping time.

Definition 2.1.2. We say that the *hypothesis* (H) is satisfied by filtrations $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$ if any $(\mathbb{P}, \mathbb{F})$-martingale is also a $(\mathbb{P}, \mathbb{K})$-martingale. We sometimes write $\mathbb{F} \hookrightarrow \mathbb{K}$ or say that $\mathbb{F}$ is *immersed* in $\mathbb{K}$.

Lemma 2.1.1. *Assume that* $\mathbb{F} \subset \mathbb{K}$. *Then* $\mathbb{F} \hookrightarrow \mathbb{K}$ *if and only if any of the following equivalent conditions holds:*
(i) for any $t \in \mathbb{R}_+$, *the* σ*-fields* $\mathcal{F}_\infty$ *and* $\mathcal{K}_t$ *are conditionally independent given* $\mathcal{F}_t$ *under* $\mathbb{P}$, *that is, for any bounded,* $\mathcal{F}_\infty$*-measurable random variable* ξ *and any bounded,* $\mathcal{K}_t$*-measurable random variable* η *we have*

$$\mathbb{E}_{\mathbb{P}}\left(\xi\eta \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\xi \,\middle|\, \mathcal{F}_t\right)\mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_t\right) \tag{2.1}$$

(ii) for any $t \in \mathbb{R}_+$ and any $u \geq t$, the σ-fields $\mathcal{F}_u$ and $\mathcal{K}_t$ are conditionally independent given $\mathcal{F}_t$,
(iii) for any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ

$$\mathbb{E}_{\mathbb{P}}\left(\xi \,\middle|\, \mathcal{K}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\xi \,\middle|\, \mathcal{F}_t\right), \tag{2.2}$$

(iv) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{K}_t$-measurable random variable η

$$\mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_\infty\right). \tag{2.3}$$

(v) In particular, if $\mathbb{K}$ is the progressive enlargement of $\mathbb{F}$ (see Definition 2.3.1) with τ, then for any $t \in \mathbb{R}_+$

$$\mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_\infty\right). \tag{2.4}$$

Definition 2.1.3. We say that the *hypothesis* (H') is satisfied by filtrations $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$ if any $(\mathbb{P}, \mathbb{F})$-semimartingale is also a $(\mathbb{P}, \mathbb{K})$-semimartingale.

The study of the hypothesis (H') is usually conducted in two steps. First, one aims to established whether the hypothesis (H') is satisfied by $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$. If the hypothesis (H') holds true, then the next step is to find out what is the $(\mathbb{P}, \mathbb{K})$-semimartingale decomposition of any $(\mathbb{P}, \mathbb{F})$-local martingale. Therefore it is important to know characterisation of the class of semimartingales given an arbitrary filtration $\mathbb{F}$. For this purpose, we quote the Bichteler-Dellacherie theorem, which shall be useful in showing that the hypothesis (H') holds between $\mathbb{F}$ and $\mathbb{K}$ (some enlargements of $\mathbb{F}$).

Theorem 2.1.1. *A $\mathbb{F}$-adapted process X is a $\mathbb{F}$-semimartingale if and only if for any sequence of simple $\mathbb{F}$-predictable process converging uniformly to zero, the stochastic integral with respect to X converges to zero in measure.*

The next proposition shows that if the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{K}$, then the hypothesis (H') is also satisfied between $\mathbb{F}$ and $\mathbb{K}_+$. (the right continuous modification of $\mathbb{K}$).

Proposition 2.1.1. *If the hypothesis (H') is satisfied between $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{K} = (\mathcal{K}_t)_{t \geq 0}$ then the hypothesis (H') is also satisfied between $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{K}_+ = (\mathcal{K}_{t+})_{t \geq 0}$.*

Proof. Suppose $M^{\mathbb{F}}$ is a bounded $\mathbb{F}$-martingale (the right continuous version). The hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{K}$ with the $\mathbb{K}$-semimartingale decomposition of $M^{\mathbb{F}}$ is given by $M^{\mathbb{F}}_t = M^{\mathbb{K}}_t + A^{\mathbb{K}}_t$, where $M^{\mathbb{K}}$ is a $\mathbb{K}$-local martingale and $A^{\mathbb{K}}$ a $\mathbb{K}$-adapted locally bounded variation process. Let $(T_n)_{n \in \mathbb{N}}$ be a common localisation sequence for the $M^{\mathbb{K}}$ and $A^{\mathbb{K}}$, then for every $n \in \mathbb{N}$, the process $M^{\mathbb{K}}_{t \wedge T_n} = M^{\mathbb{F}}_{t \wedge T_n} - A^{\mathbb{K}}_{t \wedge T_n}$ is a $\mathbb{K}$-martingale. For every $n \in \mathbb{N}$, using Proposition 2.44 from [48], we define the $\mathbb{K}_+$-martingale

$$\widehat{M}^n_t := \lim_{s \downarrow t} M^{\mathbb{K}}_{s \wedge T_n} = M^{\mathbb{F}}_{t \wedge T_n} - \lim_{s \downarrow t} A^{\mathbb{K}}_{s \wedge T_n},$$

with the second equality holds by right continuity of $M^{\mathbb{F}}$. It is also not hard to see that for every $n \in \mathbb{N}$, we have $\widehat{M}^n_t = \widehat{M}^n_{t \wedge T_n}$. One can then define a $\mathbb{K}_+$-local martingale by setting $(T_0 = 0)$

$$\widehat{M}^{\mathbb{K}_+}_t := \sum_{i=1}^{\infty} \widehat{M}^i_t \mathbb{1}_{[\![T_{i-1}, T_i [\![}(t) = M^{\mathbb{F}}_t - \sum_{i=1}^{\infty} \lim_{s \downarrow t} A^{\mathbb{K}}_{s \wedge T_i} \mathbb{1}_{[\![T_{i-1}, T_i [\![}(t).$$

This tells us that $\mathbb{K}_+$-semimartingale decomposition of any bounded $\mathbb{F}$-martingale $M^{\mathbb{F}}$ is given by $M^{\mathbb{F}}_t = \widehat{M}^{\mathbb{K}_+}_t + \widehat{A}^{\mathbb{K}_+}_t$, where the $\mathbb{K}_+$-adapted locally bounded variation process $\widehat{A}^{\mathbb{K}_+}$ is given by

$$\widehat{A}^{\mathbb{K}_+}_t := \sum_{i=1}^{\infty} \lim_{s \downarrow t} A^{\mathbb{K}}_{s \wedge T_i} \mathbb{1}_{[\![T_{i-1}, T_i [\![}(t).$$

With this we conclude the proof. $\square$

Remark 2.1.1. One should be careful that regularisation is done after the process is stopped. In general, one can not interchange stopping and limit from the right as the processes are not assumed to be cádlág.

In the following sections, we introduce the two forms of enlargements, which were studied in the existing literature, that is, the initial and the progressive enlargement of $\mathbb{F}$ with a random time τ. In addition, we shall also mention the class of honest times. To conclude this chapter, we give an overview of known results concerning semimartingale decomposition of $\mathbb{F}$-martingales in the respective enlargement of $\mathbb{F}$.

2.2 Initial Enlargement

Definition 2.2.1. The *initial enlargement* of $\mathbb{F}$ is the filtration $\mathbb{F}^{(\tau)} = (\mathcal{F}_t^{(\tau)})_{t\in\mathbb{R}_+}$ given by the equality $\mathcal{F}_t^{(\tau)} = \cap_{s>t}\left(\sigma(\tau)\vee\mathcal{F}_s\right)$ for all $t\in\mathbb{R}_+$.

Lemma 2.2.1. *Suppose that X is an integrable $\mathbb{F}^{(\tau)}$-adapted process. Then X takes the form*

$$X_t = \lim_{q\downarrow t} X^*_{\tau,q},$$

where the map $X^*_{s,q}$ is $\mathcal{B}(\bar{\mathbb{R}}_+)\otimes\mathcal{F}_q$ measurable.

Proof. By Corollary 2.4 in [97], for an integrable process X,

$$X_t = \mathbb{E}_{\mathbb{P}}\left(X_t \,\middle|\, \mathcal{F}_t^{(\tau)}\right) = \lim_{q\downarrow t}\mathbb{E}_{\mathbb{P}}\left(X_t \,\middle|\, \sigma(\tau)\vee\mathcal{F}_q\right)$$

Since for $q > t$, we have $\mathbb{E}_{\mathbb{P}}\left(X_t \,\middle|\, \sigma(\tau)\vee\mathcal{F}_q\right)\in\mathcal{F}_q\vee\sigma(\tau)$, therefore, there exists an $\mathcal{B}(\bar{\mathbb{R}}_+)\otimes\mathcal{F}_q$ measurable map $(s,\omega)\to X^*_{s,q}(\omega)$ such that

$$\lim_{q\downarrow t}\mathbb{E}_{\mathbb{P}}\left(X_t \,\middle|\, \sigma(\tau)\vee\mathcal{F}_q\right) = \lim_{q\downarrow t} X^*_{\tau,q}$$

and this concludes the proof □

Lemma 2.2.2. *Suppose X is a bounded $\mathbb{F}^{(\tau)}$-predictable process zero at $t=0$, then for any $t>0$, the process X has the following representation $X_t = X_{\tau,t}$, where $X_{s,t}\in\mathcal{B}(\bar{\mathbb{R}}_+)\otimes\mathcal{P}(\mathbb{F})$.*

Proof. The $\mathbb{F}^{(\tau)}$-predictable σ-algebra $\mathcal{P}(\mathbb{F}^{(\tau)})$ is generated by

$$P(\mathbb{F}^{(\tau)}) = \{A\times\{0\}\,:\,A\in\mathcal{F}_0^{(\tau)}\}\cup\{A\times]s,t]\ :0<s<t,\ s,t\in\mathbb{Q}_+,\ A\in\bigvee_{r<s}\mathcal{F}_r^{(\tau)}\}$$

Since the σ-algebra $\bigvee_{r<s}\mathcal{F}_r^{(\tau)}$ is contained in $\sigma(\tau)\vee\mathcal{F}_s$. Therefore, any $\bigvee_{r<s}\mathcal{F}_r^{(\tau)}$ measurable random variable A takes the form $A_{\tau,s}$, where $A_{u,s}$ is $\mathcal{B}(\bar{\mathbb{R}}_+)\otimes\mathcal{F}_s$ measurable.

We need only to prove the claim of the lemma for the generators of $\mathcal{P}(\mathbb{F}^{(\tau)})$ restricted to $(0,\infty)$. The generator thus takes the form $A_{\tau,s}\mathbb{1}_{]s,t]}$. It is obvious that $\mathbb{1}_{]s,t]}$ is a left-continuous process in t. It remains to observe that for a fixed $u\geq 0$ the map $A_{u,s}$ is $\mathcal{F}_s$-measurable. Hence $A_{u,s}\mathbb{1}_{]s,t]}$ is $\mathcal{B}(\bar{\mathbb{R}}_+)\otimes\mathcal{P}(\mathbb{F})$ measurable and it is sufficient to take $\bar{X}_{u,t} := A_{u,s}\mathbb{1}_{]s,t]}$. □

Theorem 2.2.1. *(Jacod's Criterion) If the $\mathbb{F}$-conditional distribution of a random time τ is absolutely continuous with respect to an σ-finite measure on $\mathcal{B}(\bar{\mathbb{R}}_+)$, then hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^{(\tau)}$.*

Jacod's criterion was first formulated for the initial enlargement $\mathbb{F}^{(\tau)}$ only. However, in general, one can formulate Jacod's criterion as an assumption on the $\mathbb{F}$-conditional distribution of τ and can study any enlargement of $\mathbb{F}$.

Definition 2.2.2. A pair of $(\tau, \mathbb{P})$ is said to satisfy the *density hypothesis* if there exists a positive random field $m_{s,t}$ and a σ-finite measure η on $\mathcal{B}(\bar{\mathbb{R}}_+)$ such that for a fixed $s \geq 0$, the process $(m_{s,t})_{t\geq s}$ is a $(\mathbb{P}, \mathbb{F})$-martingales and the $\mathbb{F}$-conditional distribution $F^\tau_{u,t}$ admits the following representation

$$F^\tau_{u,t} = \int_{[0,u]} m_{s,t}\, d\eta_s, \qquad u \leq t.$$

Semimartingale decomposition of $\mathbb{F}$-martingale under the density hypothesis has been examined by several papers including, Jeanblanc and Le Cam [57] and El Karoui et al. [35].

Theorem 2.2.2. *Suppose that X is a $(\mathbb{P}, \mathbb{F})$-local martingale and τ satisfies the density hypothesis. Then the process $\bar{X}$, which is given by*

$$\bar{X}_t = X_t + \int_{(0,t]} \frac{1}{m_{u,s-}} \langle X, m_{u,\cdot}\rangle_s \Big|_{u=\tau},$$

is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-local martingale.

Remark 2.2.1. Jeanblanc and Le Cam [57] have shown that if the random time satisfies density hypothesis then hypothesis (H') is also satisfied between $\mathbb{F}$ and the progressive enlargement of $\mathbb{F}$ with τ. They also computed explicitly (see Theorem 2.3.2) the semimartingale decomposition of any $(\mathbb{P}, \mathbb{F})$-martingale in the progressive enlargement of $\mathbb{F}$.

In the following, we provide a simple extension of Jacod's criterion.

Proposition 2.2.1. *Given a random time τ, if the $\mathbb{F}$-conditional distribution of τ satisfied the random Lipschitz condition, then the hypothesis (H') is satisfied between $\mathbb{F}$ and the initial enlargement of $\mathbb{F}$ with τ.*

Proof. Given an $\mathbb{F}$-semimartingale X, we will prove the claim by contradiction. If X is not a $\mathbb{F}^{(\tau)}$ semimartingale, then by Theorem 2.1.1 there exists $t \geq 0$, $\epsilon > 0$ and a sequence of simple $\mathbb{F}^{(\tau)}$-predictable process $\{\xi^n\}_{n\in\mathbb{N}}$ converging uniformly to zero, such that

$$\inf_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left(1 \wedge \left|(\xi^n \bullet X)_t\right|\right) \geq \epsilon.$$

Firstly, for large enough n, the process $(\xi^n_s)_{s\geq 0}$ is bounded and $\mathbb{F}^{(\tau)}$-predictable.

The non-decreasing process $(F_t(u))_{u\geq 0}$ is random Lipschitz, that is, there exists some positive integrable process K, such that

$$|F_t(u) - F_t(s)| \leq K_t|\eta_u - \eta_s|$$

where η is a σ-finite measure on $\mathcal{B}(\mathbb{R}_+)$. Then it is easy to see that

$$|F_t(u) - F_t(s)| \leq \sqrt{|K_t\eta_u - K_t\eta_s|}\sqrt{|F_t(u) - F_t(s)|}.$$

Then by the Kunita-Watanabe inequality, we obtain

$$\begin{aligned}
\int_{[0,\infty[} 1 \wedge (\xi^n(u) \bullet X)_t\, F_t(du) &\leq \left(\int_{[0,\infty[} |1 \wedge (\xi^n(u) \bullet X)\,|^2 K_t\, d\eta_u\right)^{1/2} \left(\int_{[0,\infty[} dF_{u,t}\right)^{1/2} \\
&\leq \left(\int_{[0,\infty[} |1 \wedge (\xi^n(u) \bullet X)\,| K_t\, d\eta_u\right)^{1/2} (F_{\infty,t} - F_{0,t})^{1/2}
\end{aligned}$$

By taking the expectation and using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(1 \wedge (\xi^n \bullet X)\right) &\leq \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty[} |1 \wedge (\xi^n(u) \bullet X)\,| K_t\, d\eta_u\right) \mathbb{E}_{\mathbb{P}}\left(F_{\infty,t} - F_{0,t}\right) \\
&\leq \mathbb{P}(\tau > 0)\left(\int_{[0,\infty[} \mathbb{E}_{\mathbb{Q}_t}\left(|1 \wedge (\xi^n(u) \bullet X)\,|\right) d\eta_u\right)
\end{aligned}$$

where $d\mathbb{Q}_t = K_t\,d\mathbb{P}$. The term $\mathbb{E}_{\mathbb{Q}_t}(|1 \wedge (\xi^n(u) \bullet X)|)$ in the right-hand side converges to zero, since the semimartingale property is invariant under an absolutely change of measure and X is an $\mathbb{F}$-semimartingale. □

2.3 Progressive Enlargement

The initial enlargement does not seem to be well suited for the analysis of a random time since it postulates that $\sigma(\tau) \subset \mathcal{F}_0^{(\tau)}$, meaning that all the information about τ is already available at time 0. It appears that the following notion of the progressive enlargement is more suitable for formulating and solving problems associated with an additional information conveyed by observations of occurrence of a random time τ.

Definition 2.3.1. *The progressive enlargement of $\mathbb{F}$ is the minimal enlargement, that is, the smallest filtration $\mathbb{F}^\tau = (\mathcal{F}^\tau)_{t\in\mathbb{R}_+}$, satisfying the usual conditions, such that $\mathbb{F} \subset \mathbb{F}^\tau$ and τ is a $\mathbb{F}^\tau$-stopping time. More explicitly, $\mathcal{F}_t^\tau = \cap_{s>t}\,\sigma(\tau \wedge t) \vee \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.*

We shall often denote the progressive enlargement of $\mathbb{F}$ with a random time τ by $\mathbb{G}$ when there is no confusion between random times.

Let $\mathbb{H}$ be the filtration generated by the process $H_t = \mathbb{1}_{\{\tau\le t\}}$. It is clear that $\mathbb{F}^\tau \subset \mathbb{F}\vee\mathbb{H} \subset \mathbb{F}^{(\tau)}$. In fact, the inclusion $\mathbb{F}^\tau \subset \mathbb{K}$ necessarily holds for any enlargement of $\mathbb{F}$. In the rest of the thesis, we will mainly work with the progressive enlargement $\mathbb{F}^\tau$. However, we would like first to clarify the relationships between various filtrations encountered in the existing literature.

One can find a comment by Meyer [86], where he essentially introduces the progressive enlargement as in the form given in Definition 2.3.2. We first make this comment explicit and find it convenient to introduce the following definition, which hinges on a natural modification of $\widetilde{\mathbb{F}}'$ which is introduced in the next section, where we study the class of honest times.

Definition 2.3.2. The family $\widehat{\mathbb{F}}^\tau = (\widehat{\mathcal{F}}_t^\tau)_{t\in\mathbb{R}_+}$ is defined by setting, for every $t \ge 0$,

$$\widehat{\mathcal{F}}_t^\tau = \{A \in \mathcal{F}^\tau \,|\, \exists\, \widetilde{A}_t \in \mathcal{F}_t \text{ and } \widehat{A}_{\tau,t} \in \mathcal{F}_t^{(\tau)} \text{ such that } A = (\widetilde{A}_t \cap \{\tau > t\}) \cup (\widehat{A}_{\tau,t} \cap \{\tau \le t\})\}.$$

We note that, for all $t \in \mathbb{R}_+$,

$$\widehat{\mathcal{F}}_t^\tau \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}, \quad \widehat{\mathcal{F}}_t^\tau \cap \{\tau \le t\} = \mathcal{F}_t^{(\tau)} \cap \{\tau \le t\}. \tag{2.5}$$

It is easily seen that the σ-field $\widehat{\mathcal{F}}_t^\tau$ is uniquely characterized by conditions (2.5). The next result shows that the family $\widehat{\mathbb{F}}^\tau$ coincides in fact with the progressive enlargement $\mathbb{F}^\tau$.

Lemma 2.3.1. *If τ is any random time then $\mathbb{F}^\tau = \widehat{\mathbb{F}}^\tau$.*

Proof. Recall that $\mathcal{F}_t^\tau = \cap_{s>t}(\sigma(\tau\wedge s) \vee \mathcal{F}_s)$ and $\mathcal{F}_t^{(\tau)} = \cap_{s>t}(\sigma(\tau) \vee \mathcal{F}_s)$ (see Definition 2.2.1 and Definition 2.3.1). To show that $\widehat{\mathcal{F}}_t^\tau = \mathcal{F}_t^\tau$, it suffices to check that conditions (2.5) are satisfied by $\mathcal{F}_t^\tau$. The following relationship for all $t \in \mathbb{R}_+$ is immediate,

$$\mathcal{F}_t \cap \{\tau > t\} \subset \mathcal{F}_t^\tau \cap \{\tau > t\} \subset \widehat{\mathcal{F}}_t^\tau \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}.$$

This shows that $\mathcal{F}_t^\tau \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}$, while on the other hand,

$$\mathcal{F}_t^\tau \cap \{\tau \le t\} = \cap_{s>t}(\sigma(\tau\wedge s) \vee \mathcal{F}_s) \cap \{\tau \le t\} = \cap_{s>t}(\sigma(\tau) \vee \mathcal{F}_s) \cap \{\tau \le t\} = \mathcal{F}_t^{(\tau)} \cap \{\tau \le t\}$$

since $\sigma(\tau\wedge s) \cap \{\tau \le t\} = \sigma(\tau) \cap \{\tau \le t\}$ for every $s > t$. □

Definition 2.3.3. We say that an enlargement $\mathbb{K}$ is *admissible before* τ if the equality $\mathcal{K}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}$ holds for every $t \in \mathbb{R}_+$.

It is easy to see that the filtration $\widehat{\mathbb{F}}^\tau$ (and hence $\mathbb{F}^\tau$) is admissible before τ. When dealing with a semimartingale decomposition of an $\mathbb{F}$-martingale after τ, we will use the following concept.

Definition 2.3.4. We say that an enlargement $\mathbb{K}$ is *admissible after* τ if the equality $\mathcal{K}_t \cap \{\tau \leq t\} = \mathcal{F}_t^{(\tau)} \cap \{\tau \leq t\}$ holds for every $t \in \mathbb{R}_+$.

It is clear that the filtration $\widehat{\mathbb{F}}^\tau$ (and thus $\mathbb{F}^\tau$) is admissible after τ for any random time. Note also that if an enlargement $\mathbb{K}$ is admissible before and after τ then simply $\mathbb{K} = \mathbb{F}^\tau$.

Lemma 2.3.2. *For any integrable, random variable η and any enlargement $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$ admissible before τ we have that, for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau>t\}}\eta \,\middle|\, \mathcal{K}_t\right) = G_t^{-1}\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau>t\}}\eta \,\middle|\, \mathcal{F}_t\right). \tag{2.6}$$

Lemma 2.3.3. *For any integrable, random variable η and any enlargement $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$ admissible after τ we have that, for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\leq t\}}\eta \,\middle|\, \mathcal{K}_t\right) = \lim_{s\downarrow t} \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\leq t\}}\eta \,\middle|\, \sigma(\tau)\vee\mathcal{F}_s\right). \tag{2.7}$$

Proof. It suffices to show that

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\leq t\}}\eta \,\middle|\, \mathcal{K}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\leq t\}}\eta \,\middle|\, \mathcal{F}_t^{(\tau)}\right). \tag{2.8}$$

Equation (2.7) will then follow from Corollary 2.4 in [97], since $\mathcal{F}_t^{(\tau)} = \cap_{s>t}\left(\sigma(\tau)\vee\mathcal{F}_s\right)$. To establish (2.8), we will first check that, for every $A \in \mathcal{K}_t$,

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_A\mathbb{1}_{\{\tau\leq t\}}\eta\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_A\mathbb{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_t^{(\tau)}\right)\right).$$

Since, by assumption, $\mathcal{K}_t \cap \{\tau \leq t\} = \mathcal{F}_t^{(\tau)} \cap \{\tau \leq t\}$, there exists an event $B \in \mathcal{F}_t^{(\tau)}$ such that $A \cap \{\tau \leq t\} = B \cap \{\tau \leq t\}$. Consequently,

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_A\mathbb{1}_{\{\tau\leq t\}}\eta\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B\mathbb{1}_{\{\tau\leq t\}}\eta\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B\mathbb{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t^{(\tau)})\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_A\mathbb{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t^{(\tau)})\big).$$

Hence

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq t\}}\eta \,|\, \mathcal{K}_t) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t^{(\tau)}) \,|\, \mathcal{K}_t\big) = \mathbb{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t^{(\tau)}),$$

since the random variable $\mathbb{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_t^{(\tau)}\right)$ is $\mathcal{K}_t$-measurable. □

Lemma 2.3.4. *Suppose X_t is an integrable $\mathbb{F}^\tau$-adapted process, the process X can be represented as*

$$X_t = \widetilde{X}_t\mathbb{1}_{\{\tau>t\}} + \lim_{q\downarrow t} X^*_{\tau,q}\mathbb{1}_{\{\tau\leq t\}},$$

where $\widetilde{X}_t = G_t^{-1}\mathbb{E}_{\mathbb{P}}\left(X_t\mathbb{1}_{\{\tau>t\}} \,\middle|\, \mathcal{F}_t\right)$ is $\mathcal{F}_t$ measurable and the map $X^*_{s,q}$ is $\mathcal{B}(\bar{\mathbb{R}}_+)\otimes\mathcal{F}_q$ measurable.

Proof. By Lemma 2.3.2 and Lemma 2.3.3

$$\begin{aligned} X_t &= \mathbb{E}_{\mathbb{P}}\left(X_t\mathbb{1}_{\{\tau>t\}} \,\middle|\, \mathcal{F}_t^\tau\right) + \mathbb{E}_{\mathbb{P}}\left(X_t\mathbb{1}_{\{\tau\leq t\}} \,\middle|\, \mathcal{F}_t^\tau\right) \\ &= \widetilde{X}_t\mathbb{1}_{\{\tau>t\}} + \lim_{q\downarrow t}\mathbb{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}\left(X_t\mathbb{1}_{\{\tau\leq t\}} \,\middle|\, \sigma(\tau)\vee\mathcal{F}_q\right) \end{aligned}$$

where $\widetilde{X}_t = G_t^{-1}\mathbb{E}_{\mathbb{P}}\left(X_t\mathbb{1}_{\{\tau>t\}} \,\middle|\, \mathcal{F}_t\right) \in \mathcal{F}_t$. Since for $q > t$, the process $\mathbb{E}_{\mathbb{P}}\left(X_t\mathbb{1}_{\{\tau\leq t\}} \,\middle|\, \sigma(\tau)\vee\mathcal{F}_q\right)$ is $\mathcal{F}_q\vee\sigma(\tau)$-measurable. Hence there exists a $\mathcal{B}(\bar{\mathbb{R}}_+)\otimes\mathcal{F}_q$-measurable map $(s,\omega)\to X^*_{s,q}(\omega)$ such that

$$\lim_{q\downarrow t}\mathbb{E}_{\mathbb{P}}\left(X_t\mathbb{1}_{\{\tau\leq t\}} \,\middle|\, \sigma(\tau)\vee\mathcal{F}_q\right) = \lim_{q\downarrow t} X^*_{\tau,q}.$$

□

Corollary 2.3.1. *The σ-algebra $\mathcal{F}_0^\tau$ is generated by $\mathcal{F}_0$ and the set $\{\tau = 0\}$.*

Proof. It is enough to apply Lemma 2.3.4 and see that, for every $A \in \mathcal{F}_0^\tau$,

$$\mathbb{1}_A = \widetilde{X}_0 \mathbb{1}_{\{\tau>0\}} + \lim_{q\downarrow 0} X_{q,0} \mathbb{1}_{\{\tau\leq 0\}}.$$

The claim follows once we notice $\widetilde{X}_0 \in \mathcal{F}_0$ and $\lim_{q\downarrow 0} X_{q,0} \in \mathcal{F}_0$, since the filtration $\mathbb{F}$ is assumed to be right-continuous. □

Lemma 2.3.5. *Suppose that X is a bounded $\mathbb{F}^\tau$-predictable process zero at $t = 0$. Then the process X has the following representation, for any $t > 0$,*

$$X_t = X'_t \mathbb{1}_{\{\tau\geq t\}} + X_{\tau,t} \mathbb{1}_{\{\tau<t\}},$$

where $X'_t \in \mathcal{P}(\mathbb{F})$ and $X_{\tau,t} \in \mathcal{B}(\bar{\mathbb{R}}_+) \otimes \mathcal{P}(\mathbb{F})$.

Proof. The $\mathbb{F}^\tau$-predictable σ-algebra $\mathcal{P}(\mathbb{F}^\tau)$ is generated by

$$P(\mathbb{F}^\tau) = \{A \times \{0\} \,:\, A \in \mathcal{F}_0\} \cup \{A \times]s,t] \,:\, 0 < s < t,\ s,t \in \mathbb{Q}_+,\ A \in \bigvee_{r<s} \mathcal{F}_r^\tau\}.$$

The σ-algebra $\bigvee_{r<s} \mathcal{F}_r^\tau$ is contained in $\sigma(\tau \wedge s) \vee \mathcal{F}_s$. Therefore, any $\bigvee_{r<s} \mathcal{F}_r^\tau$-measurable random variable A takes the form $A_{s\wedge\tau,s}$, where $A_{u,s}$ is a $\mathcal{B}(\bar{\mathbb{R}}_+) \otimes \mathcal{F}_s$-measurable map. We need only to prove the claim of the lemma on the generators of $\mathcal{P}(\mathbb{F}^\tau)$. The generator of $\mathcal{P}(\mathbb{F}^\tau)$ takes the form

$$A_{s\wedge\tau,s} \mathbb{1}_{]s,t]} = A_{s,s} \mathbb{1}_{]s,t]} \mathbb{1}_{\{\tau>t\}} + A_{s\wedge\tau,s} \mathbb{1}_{]s,t]} \mathbb{1}_{\{\tau\leq t\}}$$

It is obvious that $\mathbb{1}_{]s,t]}$ is a left-continuous process in t. It now remains to show that the process $A_{s,s}$ is $\mathbb{F}$-adapted, which implies that $A_{s,s}\mathbb{1}_{]s,t]} \in \mathcal{P}(\mathbb{F})$. On the other hand, for a fixed $u \geq 0$, the map $A_{u,s}$ is $\mathcal{F}_s$-measurable and $A_{u,s}\mathbb{1}_{]s,t]}$ is $\mathcal{B}(\bar{\mathbb{R}}_+) \otimes \mathcal{P}(\mathbb{F})$-measurable. It is sufficient to take $X'_t := A_{s,s}\mathbb{1}_{]s,t]}$ and $X_{u,t} := A_{u\wedge s,s}\mathbb{1}_{]s,t]}$. □

Theorem 2.3.1. *Let $\mathbb{F}^\tau$ be the progressive enlargement of $\mathbb{F}$ with an arbitrary random time τ. If X is a $(\mathbb{P},\mathbb{F})$-local martingale then the stopped process X^τ is a $(\mathbb{P},\mathbb{F}^\tau)$-special semimartingale.*
(i) The process

$$X_{t\wedge\tau} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d(\langle X, M\rangle_u + \check{X}^p_u) \tag{2.9}$$

is a $(\mathbb{P},\mathbb{F}^\tau)$-local martingale, where $\check{X}^p$ stands for the dual $\mathbb{F}$-predictable projection of the process $\check{X}_t = \Delta X_\tau \mathbb{1}_{\{\tau\leq t\}}$.
(ii) The process

$$X_{t\wedge\tau} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\langle X, \bar{M}\rangle_u \tag{2.10}$$

is a $(\mathbb{P},\mathbb{F}^\tau)$-local martingale, where $\bar{M}$ is the unique BMO martingale such that $\mathbb{E}_{\mathbb{P}}(N_\tau) = \mathbb{E}_{\mathbb{P}}(N_\infty \bar{M}_\infty)$ for every bounded $(\mathbb{P},\mathbb{F})$-martingale N.

Proof. See Proposition 2.5.1 and Proposition 2.5.2, respectively. □

In the recent paper by Jeanblanc and Le Cam [57], the authors computed the $(\mathbb{P},\mathbb{F}^\tau)$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-local martingale under the density hypothesis. We quote the result in the following.

Theorem 2.3.2 (Jeanblanc and Le Cam [57])**.** *Assume that τ satisfies the density hypothesis and X is a $(\mathbb{P},\mathbb{F})$-local martingale. Then the process*

$$X_t^{\mathbb{F}^\tau} = X_t - \int_{(0,t\wedge\tau]} \frac{1}{G_{s-}} d\widetilde{C}_s + \int_{(\tau,t\vee\tau]} \frac{1}{m_{u,s-}} \langle X, m_{u,\cdot}\rangle_s \Big|_{u=\tau}$$

with $\widetilde{C} = \langle X, M\rangle + B^p$ is a $(\mathbb{P},\mathbb{F}^\tau)$-local martingale.

2.4 Honest Times

In the theory of enlargement of filtrations, an important class of random times one often encounters are *honest times*. Honest times are interesting and important, because they are the earliest class of random time for which the (canonical) semimartingale decomposition in the progressive enlargement of $\mathbb{F}$ was derived. Let us remind the reader once again the definition of an honest time.

Definition 2.4.1. *A positive random variable L defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called an honest time if, for every $t > 0$, there exists an $\mathcal{F}_t$-measurable random variable L_t such that L is equal to L_t on the event $\{L \le t\}$, that is, $L\mathbb{1}_{\{L\le t\}} = L_t\mathbb{1}_{\{L\le t\}}$.*

Lemma 2.4.1. [Yor [107]] *The following properties are equivalent:*
(i) L is an honest time,
(ii) for every $u < s$, there exists an event $A_{us} \in \mathcal{F}_s$ such that $\{L \le u\} = A_{us} \cap \{L \le s\}$.

Definition 2.4.2. We define the families $\check{\mathbb{G}} = (\check{\mathcal{G}}_t)_{t\in\mathbb{R}_+}, \widetilde{\mathbb{G}} = (\widetilde{\mathcal{G}}_t)_{t\in\mathbb{R}_+}$ and $\widetilde{\mathbb{F}}^L = (\widetilde{\mathcal{F}}^L_t)_{t\in\mathbb{R}_+}$ of σ-algebras by setting, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
\check{\mathcal{G}}_t &= \{A \in \mathcal{F} \,|\, \exists\, \widetilde{A}_t \in \mathcal{F}_t \text{ such that } A \cap \{L > t\} = \widetilde{A}_t \cap \{L > t\}\},\\
\widetilde{\mathcal{G}}_t &= \big\{A \in \mathcal{F} \,|\, \exists\, \widehat{A}_t \in \mathcal{F}_t \text{ such that } A \cap \{L \le t\} = \widehat{A}_t \cap \{L \le t\}\big\},\\
\widetilde{\mathcal{F}}^L_t &= \big\{A \in \mathcal{F} \,|\, \exists\, \widetilde{A}_t, \widehat{A}_t \in \mathcal{F}_t \text{ such that } A = (\widetilde{A}_t \cap \{L > t\}) \cup (\widehat{A}_t \cap \{L \le t\})\big\} = \widetilde{\mathcal{G}}_t \cap \mathcal{G}_t.
\end{aligned}$$

Lemma 2.4.2. *If τ is a honest time then $\mathbb{F}^\tau = \widetilde{\mathbb{F}}^\tau = \widehat{\mathbb{F}}^\tau$.*

Proof. We start by noting that the inclusions $\mathbb{F}^\tau \subset \widetilde{\mathbb{F}}^\tau \subset \widehat{\mathbb{F}}^\tau$ are obviously satisfied. From Lemma 2.3.1, we know that $\mathbb{F}^\tau = \widehat{\mathbb{F}}^\tau$ for any random time, and thus $\mathbb{F}^\tau = \widetilde{\mathbb{F}}^\tau = \widehat{\mathbb{F}}^\tau$ for an honest time. □

Lemma 2.4.3. *Let L be an honest time and X be an integrable $\widetilde{\mathbb{F}}^L$-adapted process. Then the process X can be represented as follows*

$$X_t = X'_t\mathbb{1}_{\{L>t\}} + \bar{X}_t\mathbb{1}_{\{L\le t\}}$$

where both X' and $\bar{X}$ are $\mathbb{F}$-adapted.

Lemma 2.4.4. *Let L be an honest time and X be a bounded $\widetilde{\mathbb{F}}^L$-predictable process. Then the process X can be represented as follows*

$$X_t = \widetilde{X}_t\mathbb{1}_{\{L>t\}} + \widehat{X}_t\mathbb{1}_{\{L\le t\}}$$

where both $\widetilde{X}$ and $\widehat{X}$ are $\mathbb{F}$-predictable processes.

It was shown by Yor [107] that the hypothesis (H') is satisfied by $\mathbb{F}$ and $\widetilde{\mathbb{F}}^L$ (an alternative proof is given in Dellacherie and Meyer [32]). The main result in Jeulin and Yor [65] (see Theorem 2 therein) furnishes the canonical decomposition of an $\mathbb{F}$-local martingale X with respect to $\widetilde{\mathbb{F}}^L$. Recall that the filtration $\widetilde{\mathbb{F}}^L$ is well defined only in the case when a random time L is an honest time with respect to the filtration $\mathbb{F}$. For an alternative proof of this result, see Theorem 15 in Jeulin and Yor [66].

Theorem 2.4.1. *Assume that X is an $\mathbb{F}$-local martingale. We set $\widetilde{C} = \langle M, X\rangle + B^p$, where M is the martingale in the Doob-Meyer decomposition of G. Then the processes*

$$X^{\widetilde{\mathbb{F}}^L}_t = X_t - \int_{(0,t]} \frac{\mathbb{1}_{\{L\ge u\}}}{G_{u-}}\, d\left\langle \bar{M}, X\right\rangle_u + \int_{(0,t]} \frac{\mathbb{1}_{\{L<u\}}}{1 - G_{u-}}\, d\left\langle \bar{M}, X\right\rangle_u \tag{2.11}$$

is a $\widetilde{\mathbb{F}}^L$-local martingale.

Proof. See Proposition 2.5.3. □

Example 2.4.1. The last passage time of Brownian motion at particular level is an honest time.

2.5 Proofs of Semimartingale Decomposition Results

The goal of this section is to provide a very brief, but self-contained, overview of techniques used to derive the classic results regarding the semimartingale decomposition for the progressive enlargement of filtration with the special emphasis on the case of an honest time. In particular, we give here the proofs of Theorems 2.3.1 and 2.4.1. It should be stressed that neither results nor their proofs are original; for more information and further results, the interested reader is referred to Jeulin [62, 63], Jeulin and Yor [65, 66, 67] and Yor [107, 108].

Let τ be any finite random time on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We use the following notation

$$\begin{aligned} \widetilde{I} &= \mathbb{1}_{[\![0,\tau]\!]}(t), \quad \widetilde{H} = \mathbb{1}_{]\!] \tau,\infty [\![}(t), \\ I &= \mathbb{1}_{[\![0,\tau [\![}(t), \quad H = \mathbb{1}_{[\![\tau,\infty [\![}(t). \end{aligned}$$

We use the standard notation for the (dual) $\mathbb{F}$-optional and $\mathbb{F}$-predictable and projections. In particular, the dual $\mathbb{F}$-optional projection H^o (resp. the dual $\mathbb{F}$-predictable projection H^p) is a positive, increasing, $\mathbb{F}$-optional (resp. $\mathbb{F}$-predictable) process. We denote $(X \bullet Y)_t = \int_{(0,t]} X_u \, dY_u$.

Remark 2.5.1. The 'usual' notation reads: For optional projections, referring to Section 1.5, it is common to write

$$G = {}^o(1 - H) = {}^oI, \quad F = {}^oH = 1 - G.$$

It is known that ${}^p\widetilde{I} = G_-$ (see Jeulin [62]). Hence ${}^p\widetilde{H} = {}^p(1 - \widetilde{I}) = 1 - G_- = F_-$. Note that ${}^pF \neq F_-$, in general, since it may happen that ${}^pH \neq {}^p\widetilde{H}$ or, equivalently, that ${}^p\mathbb{1}_{[\![\tau]\!]} \neq 0$.

Lemma 2.5.1. *Let N be a bounded $(\mathbb{P}, \mathbb{F})$-martingale with $N_0 = 0$. Then*

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(N_\tau) &= \mathbb{E}_{\mathbb{P}}((N \bullet H)_\infty) = \mathbb{E}_{\mathbb{P}}((N \bullet H^o)_\infty) \overset{\text{Ito}}{=} \mathbb{E}_{\mathbb{P}}(N_\infty H^o_\infty) \\ &= \mathbb{E}_{\mathbb{P}}(N_\infty \bar{M}_\infty) = \mathbb{E}_{\mathbb{P}}([N, \bar{M}]_\infty) = \mathbb{E}_{\mathbb{P}}(\langle N, \bar{M} \rangle_\infty) \end{aligned} \tag{2.12}$$

where we set $\bar{M}_t = \mathbb{E}_{\mathbb{P}}(H^o_\infty \,|\, \mathcal{F}_t)$ so that $\bar{M}_\infty = H^o_\infty$.

Remark 2.5.2. One can check that ${}^oI = \bar{M} - H^o$ and thus the equality $G = \bar{M} - H^o$ is an $\mathbb{F}$-optional decomposition of G. To this end, it suffices to show that for any increasing, $\mathbb{F}$-adapted process B with $B_0 = 0$ we have that

$$\mathbb{E}_{\mathbb{P}}(((\bar{M} - H^o) \bullet B)_\infty) = \mathbb{E}_{\mathbb{P}}(({}^oI \bullet B)_\infty). \tag{2.13}$$

To show that (2.13) is valid, we observe that, on the one hand,

$$\mathbb{E}_{\mathbb{P}}(({}^oI \bullet B)_\infty) = \mathbb{E}_{\mathbb{P}}((I \bullet B)_\infty) = \mathbb{E}_{\mathbb{P}}(B_{\tau-}).$$

On the other hand, by setting $\bar{Z} = \bar{M} - H^o$, we obtain from the Itô formula (note that $\bar{Z}_\infty = 0$)

$$\mathbb{E}_{\mathbb{P}}((\bar{Z} \bullet B)_\infty) = \mathbb{E}_{\mathbb{P}}((\bar{Z}B)_\infty - (B_- \bullet \bar{M})_\infty + (B_- \bullet H^o)_\infty) = \mathbb{E}_{\mathbb{P}}((B_- \bullet H)_\infty) = \mathbb{E}_{\mathbb{P}}(B_{\tau-}).$$

We conclude that (2.13) is satisfied.

Remark 2.5.3. One can show that the $(\mathbb{P}, \mathbb{F})$-martingale $\bar{M}$ belongs to the class BMO and it is the unique BMO $(\mathbb{P}, \mathbb{F})$-martingale for which the equality $\mathbb{E}_{\mathbb{P}}(N_\tau) = \mathbb{E}_{\mathbb{P}}(\langle N, \bar{M} \rangle_\infty)$ holds for any bounded $(\mathbb{P}, \mathbb{F})$-martingale N.

Lemma 2.5.2. *Let N be a bounded $(\mathbb{P}, \mathbb{F})$-martingale with $N_0 = 0$. Then*

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(N_{\tau-}) &= \mathbb{E}_{\mathbb{P}}((N_-)_\tau) = \mathbb{E}_{\mathbb{P}}((N_- \bullet H)_\infty) = \mathbb{E}_{\mathbb{P}}((N_- \bullet H^p)_\infty) \overset{\text{Ito}}{=} \mathbb{E}_{\mathbb{P}}(N_\infty H^p_\infty) \\ &= \mathbb{E}_{\mathbb{P}}(N_\infty M_\infty) = \mathbb{E}_{\mathbb{P}}([N, M]_\infty) = \mathbb{E}_{\mathbb{P}}(\langle N, M \rangle_\infty) \end{aligned} \tag{2.14}$$

where we set $M_t = \mathbb{E}_{\mathbb{P}}(H^p_\infty \,|\, \mathcal{F}_t)$ so that $M_\infty = H^p_\infty$.

It is easy to see that the equality $G = M - A$ holds and is in fact the predictable additive decomposition of G introduced in Section 1.5.

Remark 2.5.4. One can show that the $(\mathbb{P}, \mathbb{F})$-martingale M belongs to the class BMO and it is the unique BMO $(\mathbb{P}, \mathbb{F})$-martingale for which the equality $\mathbb{E}_{\mathbb{P}}(N_{\tau-}) = \mathbb{E}_{\mathbb{P}}(\langle N, M\rangle_\infty)$ holds for any bounded $(\mathbb{P}, \mathbb{F})$-martingale N.

Remark 2.5.5. It is worth noting that we may equally well use H^o and $\bar{M}$ in Lemma 2.5.2. However, a suitable 'correction term' will appear in formula (2.14). Specifically, (2.14) will become

$$\mathbb{E}_{\mathbb{P}}(N_{\tau-}) = \mathbb{E}_{\mathbb{P}}(\langle N, \bar{M}\rangle_\infty) - \mathbb{E}_{\mathbb{P}}(\langle N, H^o\rangle_\infty).$$

Remark 2.5.6. If τ avoids $\mathbb{F}$-stopping times then $\bar{M} = M$. The avoidance property (**A**) implies that ${}^o(\mathbb{1}_{[\![\tau]\!]}) = 0$ and thus also ${}^p(\mathbb{1}_{[\![\tau]\!]}) = {}^p(H - J) = 0$. Hence ${}^pF = {}^pH = F_-$. Furthermore, $(\mathbb{1}_{[\![\tau]\!]})^o = (\mathbb{1}_{[\![\tau]\!]})^p = 0$. More details on the avoidance property (**A**) are required. Also, one can study the property (**C**) stating that all $(\mathbb{P}, \mathbb{F})$-local martingales are continuous.

2.5.1 Stopped Processes: Arbitrary Random Times

Let $\mathbb{K}$ be any filtration, for instance, any enlargement of $\mathbb{F}$. The following result is well known.

Lemma 2.5.3. *A bounded $\mathbb{K}$-semimartingale Y is a $\mathbb{K}$-martingale if and only if for any bounded $\mathbb{K}$-predictable process X we have that*

$$\mathbb{E}_{\mathbb{P}}((X \bullet Y)_\infty) = 0.$$

In the rest of this section, we assume that $\mathbb{K}$ is any enlargement of $\mathbb{F}$ such that τ is a $\mathbb{K}$-stopping time and $\mathbb{K}$ is admissible prior to τ. Recall that $\mathbb{K}$ is *admissible prior to* τ if for every $t \in \mathbb{R}_+$

$$\mathcal{K}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\},$$

that is: for every $t \in \mathbb{R}_+$ and $A_t \in \mathcal{K}_t$ there exists $\widetilde{A}_t \in \mathcal{F}_t$ such that $A_t \cap \{\tau > t\} = \widetilde{A}_t \cap \{\tau > t\}$.

Lemma 2.5.4. *For any $\mathbb{K}$-predictable process X there exists a unique $\mathbb{F}$-predictable process K such that $X\mathbb{1}_{[\![0,\tau]\!]} = K\mathbb{1}_{[\![0,\tau]\!]}$. If X is bounded then one may take K bounded by the same constant.*

Lemma 2.5.5. *Let X be a $\mathbb{K}$-predictable process. Then*

$$ {}^p(X\widetilde{I}) = {}^p(K\widetilde{I}) = {}^p\widetilde{I}K. \tag{2.15}$$

We assume that Y is a bounded $(\mathbb{P}, \mathbb{F})$-martingale and we take for granted that the stopped process Y^τ is a special $(\mathbb{P}, \mathbb{K})$-semimartingale. We only address the problem of explicit computation of the $(\mathbb{P}, \mathbb{K})$-canonical decomposition of Y^τ. Propositions 2.5.1 and 2.5.2 furnish two alternative solutions to this problem, originally due to Jeulin [63] (Proposition 4.16 in [63]) and Jeulin and Yor [65] (Theorem 1 in [65]), respectively.

Proposition 2.5.1. *Assume that Y is a bounded $(\mathbb{P}, \mathbb{F})$-martingale. The $(\mathbb{P}, \mathbb{K})$-canonical decomposition of the stopped process Y^τ reads $Y^\tau = \widehat{M} + \widehat{A}$ where the $\mathbb{K}$-predictable process of finite variation $\widehat{A}$ equals*

$$\widehat{A}_t = \Big(\frac{\widetilde{I}}{{}^p\widetilde{I}} \bullet \langle Y, \bar{M}\rangle\Big)_t = \int_0^t \frac{\widetilde{I}_u}{{}^p\widetilde{I}_u}\, d\langle Y, \bar{M}\rangle_u. \tag{2.16}$$

Proof. The proof hinges on Lemma 2.5.3. Let thus X be any bounded $\mathbb{K}$-predictable process. Recall that we denote by K the unique bounded $\mathbb{F}$-predictable process such that $X\mathbb{1}_{[\![0,\tau]\!]} = K\mathbb{1}_{[\![0,\tau]\!]}$. Hence

$$\mathbb{E}_{\mathbb{P}}((X \bullet Y^\tau)_\infty) = \mathbb{E}_{\mathbb{P}}((K \bullet Y^\tau)_\infty) = \mathbb{E}_{\mathbb{P}}((K \bullet Y)_\tau).$$

By applying Lemma 2.5.1 to the bounded $(\mathbb{P},\mathbb{F})$-martingale $N = K \bullet Y$, we obtain

$$\mathbb{E}_{\mathbb{P}}((K \bullet Y)_{\tau}) = \mathbb{E}_{\mathbb{P}}(\langle K \bullet Y, \bar{M}\rangle_{\infty}) = \mathbb{E}_{\mathbb{P}}((K \bullet \langle Y, \bar{M}\rangle)_{\infty})$$

and Lemma 2.5.5 yields

$$\mathbb{E}_{\mathbb{P}}((K \bullet \langle Y, \bar{M}\rangle)_{\infty}) = \mathbb{E}_{\mathbb{P}}\Big(\Big(K\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet \langle Y, \bar{M}\rangle\Big)_{\infty}\Big) = \mathbb{E}_{\mathbb{P}}\Big(\Big(X\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet \langle Y, \bar{M}\rangle\Big)_{\infty}\Big) = \mathbb{E}_{\mathbb{P}}((X \bullet \widehat{A})_{\infty}).$$

We conclude that for any bounded $\mathbb{K}$-predictable process X the following equality holds

$$\mathbb{E}_{\mathbb{P}}((X \bullet Y^{\tau})_{\infty}) = \mathbb{E}_{\mathbb{P}}((X \bullet \widehat{A})_{\infty}).$$

Therefore, in view of Lemma 2.5.3, the process $\widehat{M} = Y^{\tau} - \widehat{A}$ is a $\mathbb{K}$-local martingale. It is also clear that $\widehat{A}$ is a $\mathbb{K}$-predictable process of finite variation and thus the equality $Y^{\tau} = \widehat{M} + \widehat{A}$ is the $(\mathbb{P},\mathbb{K})$-canonical decomposition of the stopped process Y^{τ}. □

Proposition 2.5.2. *Assume that Y is a bounded $(\mathbb{P},\mathbb{F})$-martingale. The $(\mathbb{P},\mathbb{K})$-canonical decomposition of the stopped process Y^{τ} reads $Y^{\tau} = \widetilde{M} + \widetilde{A}$ where the $\mathbb{K}$-predictable process of finite variation $\widetilde{A}$ equals*

$$\widetilde{A}_t = \Big(\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet (\langle Y, M\rangle + \check{Y})\Big)_t = \int_0^t \frac{\widetilde{I}_u}{{}^{p}\widetilde{I}_u}\, d(\langle Y, M\rangle_u + \check{Y}_u) \tag{2.17}$$

where we denote $\check{Y} = (\Delta Y \bullet H)^p$.

Proof. Let X be any bounded $\mathbb{K}$-predictable process. As before, we denote by K the unique bounded $\mathbb{F}$-predictable process such that $X\mathbb{1}_{[\![0,\tau]\!]} = K\mathbb{1}_{[\![0,\tau]\!]}$. We note that

$$\mathbb{E}_{\mathbb{P}}((X \bullet Y^{\tau})_{\infty}) = \mathbb{E}_{\mathbb{P}}((K \bullet Y^{\tau})_{\infty}) = \mathbb{E}_{\mathbb{P}}((K \bullet Y)_{\tau-}) + \mathbb{E}_{\mathbb{P}}(K \bullet (\Delta Y \bullet H)_{\infty}).$$

By applying Lemma 2.5.2 to the bounded $(\mathbb{P},\mathbb{F})$-martingale $N = K \bullet Y$, we obtain

$$\mathbb{E}_{\mathbb{P}}((K \bullet Y)_{\tau-}) = \mathbb{E}_{\mathbb{P}}(\langle K \bullet Y, M\rangle_{\infty}) = \mathbb{E}_{\mathbb{P}}((K \bullet \langle Y, M\rangle)_{\infty})$$

and Lemma 2.5.5 gives

$$\mathbb{E}_{\mathbb{P}}((K \bullet \langle Y, M\rangle)_{\infty}) = \mathbb{E}_{\mathbb{P}}\Big(\Big(K\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet \langle Y, M\rangle\Big)_{\infty}\Big) = \mathbb{E}_{\mathbb{P}}\Big(\Big(X\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet \langle Y, M\rangle\Big)_{\infty}\Big).$$

Furthermore, since K is an $\mathbb{F}$-predictable process

$$\mathbb{E}_{\mathbb{P}}((K \bullet (\Delta Y \bullet H))_{\infty}) = \mathbb{E}_{\mathbb{P}}((K \bullet (\Delta Y \bullet H)^p)_{\infty}) = \mathbb{E}_{\mathbb{P}}\Big(\Big(X\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet \check{Y}\Big)_{\infty}\Big).$$

We conclude that for any bounded $\mathbb{K}$-predictable process X

$$\mathbb{E}_{\mathbb{P}}((X \bullet Y^{\tau})_{\infty}) = \mathbb{E}_{\mathbb{P}}\Big(\Big(X\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet \langle Y, M\rangle\Big)_{\infty}\Big) + \mathbb{E}_{\mathbb{P}}\Big(\Big(X\frac{\widetilde{I}}{{}^{p}\widetilde{I}} \bullet \check{Y}\Big)_{\infty}\Big) = \mathbb{E}_{\mathbb{P}}((X \bullet \widetilde{A})_{\infty}).$$

Hence $\widetilde{M} = Y^{\tau} - \widetilde{A}$ is a $\mathbb{K}$-local martingale. It is also clear that $\widetilde{A}$ is a $\mathbb{K}$-predictable process of finite variation. □

2.5.2 Non-Stopped Processes: Honest Times

Let τ be a finite honest time. We recall from Definition 2.4.2 that the filtration $\widetilde{\mathbb{F}}^\tau$ is given by the formula

$$\widetilde{\mathcal{F}}_t^\tau = \{ A \in \mathcal{G}_\infty \mid \exists\, \widetilde{A}_t, \widehat{A}_t \in \mathcal{F}_t \text{ such that } A = (\widetilde{A}_t \cap \{\tau > t\}) \cup (\widehat{A}_t \cap \{\tau \leq t\}) \}. \tag{2.18}$$

We know from Lemma 2.4.4 that a process X is a $\widetilde{\mathbb{F}}^\tau$-predictable process if and only if

$$X = K\widetilde{I} + L\widetilde{H} = K\mathbb{1}_{[\![0,\tau]\!]} + L\mathbb{1}_{]\!] \tau,\infty [\![}(t)$$

for some $\mathbb{F}$-predictable processes K and L. If X is bounded that one may take K and L bounded by the same constant.

Lemma 2.5.6. *Let X be a $\widetilde{\mathbb{F}}^\tau$-predictable process. Then*

$${}^p(X\widetilde{I}) = {}^p(K\widetilde{I}) = {}^p\widetilde{I}K, \quad {}^p(X\widetilde{H}) = {}^p(L\widetilde{H}) = {}^p\widetilde{H}L. \tag{2.19}$$

We are in a position to compute the $(\mathbb{P}, \widetilde{\mathbb{F}}^\tau)$-canonical decomposition of a bounded $(\mathbb{P}, \mathbb{F})$-martingale Y. We take for granted that Y is a $(\mathbb{P}, \widetilde{\mathbb{F}}^\tau)$-semimartingale. The following result can be traced back to Theorem A in Barlow [6], Theorem 5.10 in Jeulin [63], Theorem 2 in Jeulin and Yor [65], and Theorem 15 in Jeulin and Yor [66].

Proposition 2.5.3. *Assume that Y is a bounded $(\mathbb{P}, \mathbb{F})$-martingale. The $(\mathbb{P}, \widetilde{\mathbb{F}}^\tau)$-canonical decomposition of Y reads $Y = \widehat{M} + \widehat{A}$ where the $\widetilde{\mathbb{F}}^\tau$-predictable process of finite variation $\widehat{A}$ equals*

$$\widehat{A}_t = \Big(\frac{\widetilde{I}}{{}^p\widetilde{I}} \bullet \langle Y, \bar{M} \rangle \Big)_t - \Big(\frac{\widetilde{H}}{{}^p\widetilde{H}} \bullet \langle Y, \bar{M} \rangle \Big)_t = \int_0^t \frac{\widetilde{I}_u}{{}^p\widetilde{I}_u}\, d\langle Y, \bar{M} \rangle_u - \int_0^t \frac{\widetilde{H}_u}{{}^p\widetilde{H}_u}\, d\langle Y, \bar{M} \rangle_u. \tag{2.20}$$

Proof. Let X be any bounded $\widetilde{\mathbb{F}}^\tau$-predictable process. Then

$$\mathbb{E}_\mathbb{P}((X \bullet Y)_\infty) = \mathbb{E}_\mathbb{P}((K\widetilde{I} \bullet Y)_\infty) + \mathbb{E}_\mathbb{P}((L(1-\widetilde{I}) \bullet Y)_\infty) = \mathbb{E}_\mathbb{P}((K \bullet Y)_\tau) - \mathbb{E}_\mathbb{P}((L \bullet Y)_\tau)$$

since $\mathbb{E}_\mathbb{P}((L \bullet Y)_\infty) = 0$. By applying Lemma 2.5.1 to the bounded $(\mathbb{P}, \mathbb{F})$-martingales $K \bullet Y$ and $L \bullet Y$, we obtain

$$\mathbb{E}_\mathbb{P}((K \bullet Y)_\tau) = \mathbb{E}_\mathbb{P}(\langle K \bullet Y, \bar{M} \rangle_\infty) = \mathbb{E}_\mathbb{P}((K \bullet \langle Y, \bar{M} \rangle)_\infty)$$

and

$$\mathbb{E}_\mathbb{P}((L \bullet Y)_\tau) = \mathbb{E}_\mathbb{P}(\langle L \bullet Y, \bar{M} \rangle_\infty) = \mathbb{E}_\mathbb{P}((L \bullet \langle Y, \bar{M} \rangle)_\infty).$$

Lemma 2.5.6 gives

$$\mathbb{E}_\mathbb{P}((K \bullet \langle Y, \bar{M} \rangle)_\infty) = \mathbb{E}_\mathbb{P}\Big(\Big(K \frac{\widetilde{I}}{{}^p\widetilde{I}} \bullet \langle Y, \bar{M} \rangle \Big)_\infty \Big) = \mathbb{E}_\mathbb{P}\Big(\Big(X \frac{\widetilde{I}}{{}^p\widetilde{I}} \bullet \langle Y, \bar{M} \rangle \Big)_\infty \Big)$$

and

$$\mathbb{E}_\mathbb{P}((L \bullet \langle Y, \bar{M} \rangle)_\infty) = \mathbb{E}_\mathbb{P}\Big(\Big(L \frac{\widetilde{H}}{{}^p\widetilde{H}} \bullet \langle Y, \bar{M} \rangle \Big)_\infty \Big) = \mathbb{E}_\mathbb{P}\Big(\Big(X \frac{\widetilde{H}}{{}^p\widetilde{H}} \bullet \langle Y, \bar{M} \rangle \Big)_\infty \Big).$$

We conclude that for any bounded $\widetilde{\mathbb{F}}^\tau$-predictable process X the following equality holds

$$\mathbb{E}_\mathbb{P}((X \bullet Y)_\infty) = \mathbb{E}_\mathbb{P}((X \bullet \widehat{A})_\infty).$$

Hence $\widehat{M} = Y - \widehat{A}$ is a $\widetilde{\mathbb{F}}^\tau$-local martingale. It is easy to see that $\widehat{A}$ is a $\widetilde{\mathbb{F}}^\tau$-predictable process of finite variation, so that the equality $Y = \widehat{M} + \widehat{A}$ is the $(\mathbb{P}, \widetilde{\mathbb{F}}^\tau)$-canonical decomposition of Y. □

Remark 2.5.7. One can also establish a counterpart of Proposition 2.5.2.

Chapter 3

Constructing Random Times with Given Survival Processes

This chapter is based on the paper by Gapeev, Jeanblanc, Li and Rutkowski [44] published in *Contemporary Quantitative Finance*, C. Chiarella and A. Novikov, eds., Springer, Berlin, 2010, pp. 255–280.

3.1 Introduction

The goal of this work is to address the following problem:

Problem (P). Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Assume that we are given a strictly positive, càdlàg, $(\mathbb{P}, \mathbb{F})$-local martingale N with $N_0 = 1$ and an $\mathbb{F}$-adapted, continuous, increasing process Λ, with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$, and such that $G_t := N_t e^{-\Lambda_t} \le 1$ for every $t \in \mathbb{R}_+$. The goal is to construct a random time τ on an extended probability space and a probability measure $\mathbb{Q}$ on this space such that:
(i) $\mathbb{Q}$ is equal to $\mathbb{P}$ when restricted to $\mathbb{F}$, that is, $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ for every $t \in \mathbb{R}_+$,
(ii) the Azéma supermartingale $G_t^{\mathbb{Q}} := \mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)$ of τ under $\mathbb{Q}$ with respect to the filtration $\mathbb{F}$ satisfies

$$G_t^{\mathbb{Q}} = N_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{3.1}$$

In that case, the pair $(\tau, \mathbb{Q})$ is called a solution to Problem (P).

We will sometimes refer to the Azéma supermartingale $G^{\mathbb{Q}}$ as the *survival process* of τ under $\mathbb{Q}$ with respect to $\mathbb{F}$. The solution to this problem is well known if $N_t = 1$ for every $t \in \mathbb{R}_+$ (see Section 3.3) and thus we will focus in what follows on the case where N is not equal identically to 1.

Condition (i) implies that the postulated inequality $G_t := N_t e^{-\Lambda_t} \le 1$ is necessary for the existence of a solution $(\tau, \mathbb{Q})$. Note also that in view of (i), the joint distribution of (N, Λ) is set to be identical under $\mathbb{P}$ and $\mathbb{Q}$ for any solution $(\tau, \mathbb{Q})$ to Problem (P). In particular, N is not only a $(\mathbb{P}, \mathbb{F})$-local martingale, but also a $(\mathbb{Q}, \mathbb{F})$-local martingale. However, in the construction of a solution to Problem (P) provided in this work, the so-called H-hypothesis is not satisfied under $\mathbb{Q}$ by the filtration $\mathbb{F}$ and the enlarged filtration $\mathbb{G}$ generated by $\mathbb{F}$ and the observations of τ. Hence the process N is not necessarily a $(\mathbb{Q}, \mathbb{G})$-local martingale.

In the approach proposed in this work, in the first step we construct a finite random time τ on an extended probability space using the *canonical construction* in such a way that

$$G_t^{\mathbb{P}} := \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+.$$

To avoid the need for an extension of Ω, it suffices to postulate, without loss of generality, that there exists a random variable Θ defined on Ω such that Θ is exponentially distributed under $\mathbb{P}$ and it is independent of $\mathcal{F}_\infty$. In the second step, we propose a change of the probability measure by making use of a suitable version of Girsanov's theorem. Since we purportedly identify the extended space with Ω, it make sense to compare the probability measures $\mathbb{P}$ and $\mathbb{Q}$. Let us mention in this regard that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are not necessarily equivalent. However, for any solution $(\tau, \mathbb{Q})$ to Problem (P), the equality $\mathbb{Q}(\tau < \infty) = 1$ is satisfied in the present set-up (see Lemma 3.4.1) and thus τ is necessarily a finite random time under $\mathbb{Q}$.

In the existing literature, one can find easily examples where the Doob-Meyer decomposition of the Azéma supermartingale is given, namely, $G_t = M_t - A_t$ (see, e.g., Mansuy and Yor [79]). It is then straightforward to deduce the multiplicative decomposition by setting $N_t = \int_0^t e^{\Lambda_s}\, dM_s$ and $\Lambda_t = \int_0^t \frac{dA_s}{G_{s-}}$. However, to the best of our knowledge, a complete solution to the problem stated above is not yet available, though some partial results were obtained. Nikeghbali and Yor [93] study a similar problem for a particular process Λ, namely, $\Lambda_t = \ln(\sup_{s\le t} N_s)$ for a local martingale N, which converges to 0 as t goes to infinity. It is worth stressing that in [93] the process G can take the value one for some $t > 0$. We will conduct the first part of our study under the standing assumption that $G_t \le 1$. However, to provide an explicit construction of a probability measure $\mathbb{Q}$, we will work in Section 3.5 under the stronger assumption that the inequality $G_t < 1$ holds for every $t > 0$.

This chapter is organized as follows. We start by presenting in Section 3.2 an example of a random time τ, which is not a stopping time with respect to the filtration $\mathbb{F}$, such that the Azéma supermartingale of τ with respect to $\mathbb{F}$ can be computed explicitly. In fact, we revisit here a classic example arising in the non-linear filtering theory. In the present context, it can be seen as a motivation for the problem stated at the beginning. In addition, some typical features of the Azéma supermartingale, which are apparent in the filtering example, are later rediscovered in a more general set-up, which is examined in the subsequent sections.

The goal of Section 3.3 is to furnish some preliminary results on Girsanov's change of a probability measure in the general set-up. In Section 3.4, the original problem is first reformulated and then reduced to a more tractable analytical problem (see Problems (P.1)–(P.3) therein). In Section 3.5, we analyze in some detail the case of a Brownian filtration. Under the assumption that $G_t < 1$ for every $t \in \mathbb{R}_+$, we identify a solution to the original problem in terms of the Radon-Nikodým density process.

Section 3.6 discusses the relevance of the multiplicative decomposition of survival process of a default time τ for the risk-neutral valuation of credit derivatives. From this perspective, it is important to observe that a random time τ constructed in this work has the same intensity under $\mathbb{P}$ and $\mathbb{Q}$, but it has different conditional probability distributions with respect to $\mathbb{F}$ under $\mathbb{P}$ and $\mathbb{Q}$. This illustrates the important fact that the default intensity does not contain enough information to price credit derivatives (in this regard, we refer to El Karoui et al. [36]). In several papers in the financial literature, the modeling of credit risk is based on the postulate that the process

$$M_t := \mathbb{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \lambda_u\, du$$

is a martingale with respect to a filtration $\mathbb{G}$ such that τ is a (totally inaccessible) $\mathbb{G}$-stopping time. However, we will argue in Section 3.6 that this information is insufficient for the computation of prices of credit derivatives. Indeed, it appears that, except for the most simple examples of models and credit derivatives, the martingale component in the multiplicative decomposition of the Azéma supermartingale of τ has a non-negligible impact on risk-neutral values of credit derivatives.

3.2 Filtering Example

The starting point for this research was a well-known problem arising in the filtering theory. The goal of this section is to recall this example and to examine some interesting features of the conditional distributions of a random time, which will be later rediscovered in a different set-up.

3.2.1 Azéma Supermartingale

Let $W = (W_t,\, t \in \mathbb{R}_+)$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and τ be a random time, independent of W and such that $\mathbb{P}(\tau > t) = e^{-\lambda t}$ for every $t \in \mathbb{R}_+$ and some fixed $\lambda > 0$. We define the process $U = (U_t,\, t \in \mathbb{R}_+)$ by setting

$$U_t = \exp\left(\left(a + b - \frac{\sigma^2}{2}\right) t - b(t-\tau)^+ + \sigma\, W_t\right),$$

where a, b and σ are some given strictly positive constants. One can check that the process U solves the stochastic differential equation

$$dU_t = U_t \left(a + b\, \mathbb{1}_{\{\tau > t\}}\right) dt + U_t\, \sigma\, dW_t. \tag{3.2}$$

In the filtering problem, the goal is to assess the conditional probability that the moment τ has already occurred by a given date t, using the observations of the process U driven by (3.2).

Let us take as $\mathbb{F}$ the natural filtration of the process U, that is, $\mathcal{F}_t = \sigma(U_s \,|\, 0 \leq s \leq t)$ for $t \in \mathbb{R}_+$. By means of standard arguments (see, e.g., [104, Chapter IV, Section 4] or [77, Chapter IX, Section 4]), it can be shown that the process U admits the following semimartingale decomposition in its own filtration

$$dU_t = U_t(a + bG_t)\, dt + U_t \sigma\, d\bar{W}_t,$$

where $G = (G_t,\, t \in \mathbb{R}_+)$ is the Azéma supermartingale, given by $G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$, and the innovation process $\bar{W} = (\bar{W}_t,\, t \in \mathbb{R}_+)$, defined by

$$\bar{W}_t = W_t + \frac{b}{\sigma} \int_0^t \left(\mathbb{1}_{\{\tau > u\}} - G_u\right) du,$$

is a standard Brownian motion with respect to $\mathbb{F}$. It is easy to show, using the arguments based on the notion of strong solutions of stochastic differential equations (see, e.g. Liptser and Shiryaev [77, Chapter IV, Section 4]), that the natural filtration of $\bar{W}$ coincides with $\mathbb{F}$. It follows from [77, Chapter IX, Section 4] (see also Shiryaev [104, Chapter IV, Section 4]) that the process G solves the following stochastic differential equation

$$dG_t = -\lambda G_t\, dt + \frac{b}{\sigma}\, G_t(1 - G_t)\, d\bar{W}_t, \tag{3.3}$$

so that the process $N = (N_t,\, t \in \mathbb{R}_+)$, given by $N_t = e^{\lambda t} G_t$, satisfies

$$dN_t = \frac{b}{\sigma}\, e^{\lambda t}\, G_t(1 - G_t)\, d\bar{W}_t. \tag{3.4}$$

Since $G(1-G)$ is bounded, it is clear that N is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale with $N_0 = 1$. We conclude that the Azéma supermartingale G of τ with respect to the filtration $\mathbb{F}$ admits the following representation

$$G_t = N_t e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+, \tag{3.5}$$

where the $(\mathbb{P}, \mathbb{F})$-martingale N is given by (3.4).

Let us finally observe that equality (3.3) provides the (additive) Doob-Meyer decomposition of the bounded $(\mathbb{P}, \mathbb{F})$-supermartingale G, whereas equality (3.5) yields its multiplicative decomposition.

3.2.2 Conditional Distributions

From the definition of the Azéma supermartingale G and the fact that $(G_t\, e^{\lambda t},\, t \in \mathbb{R}_+)$ is a $(\mathbb{P}, \mathbb{F})$-martingale it follows that, for every fixed $u > 0$ and every $t \in [0, u]$,

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{P}(\tau > u \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t\big) = e^{-\lambda u}\, \mathbb{E}_{\mathbb{P}}(N_u \,|\, \mathcal{F}_t) = e^{-\lambda u}\, N_t. \tag{3.6}$$

Standard arguments given in Shiryaev [103, Chapter IV, Section 4] (which are also summarized in [104, Chapter IV, Section 4]), based on an application of the Bayes formula, yield the following result, which extends formula (3.6) to $t \in [u, \infty)$.

Proposition 3.2.1. *The conditional survival probability process equals, for every $t, u \in \mathbb{R}_+$,*

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = 1 - \frac{X_t}{X_{u\wedge t}} + X_t Y_{u\wedge t}\, e^{-\lambda u}, \tag{3.7}$$

where the process Y is given by

$$Y_t = \exp\left(\frac{b}{\sigma^2}\Big(\ln U_t - \frac{2a+b-\sigma^2}{2}\, t\Big)\right) \tag{3.8}$$

and the process X satisfies

$$\frac{1}{X_t} = 1 + \int_0^t e^{-\lambda u}\, dY_u. \tag{3.9}$$

It follows immediately from (3.7) that

$$G_t = X_t Y_t\, e^{-\lambda t}$$

so that the equality $N = XY$ is valid, and thus (3.7) coincides with (3.6) when $t \in [0, u]$. Moreover, by standard computations, we see that

$$dY_t = \frac{b}{\sigma}\, Y_t\, dW_t + \frac{b^2}{\sigma^2}\, G_t Y_t\, dt. \tag{3.10}$$

Using (3.4) and (3.10), we obtain

$$dX_t = d\Big(\frac{N_t}{Y_t}\Big) = -\frac{b}{\sigma}\, G_t X_t\, d\bar{W}_t,$$

and thus X is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale with $X_0 = 1$.

Finally, it is interesting to note that we deal here with the model where

$$\tau = \inf\{t \in \mathbb{R}_+ : \Lambda_t \geq \Theta\}$$

with $\Lambda_t = \lambda t$ (so that $\lambda\tau = \Theta$) and the barrier Θ is an exponentially distributed random variable, which is not independent of the σ-field $\mathcal{F}_\infty$. Indeed, we have that, for every $u > 0$ and $0 \leq t < u/\lambda$,

$$\mathbb{P}(\Theta > u \,|\, \mathcal{F}_t) = \mathbb{P}(\tau > u/\lambda \,|\, \mathcal{F}_t) = N_t e^{-u} \neq e^{-u}.$$

3.3 Preliminary Results

We start by introducing notation. Let τ be a random time, defined the probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F}$, and such that $\mathbb{P}(\tau > 0) = 1$. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ the $\mathbb{P}$-completed and right-continuous version of the progressive enlargement of the filtration $\mathbb{F}$ by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t\in\mathbb{R}_+}$ generated by the process $H_t = \mathbb{1}_{\{\tau\leq t\}}$. It is assumed throughout that the H-*hypothesis* (see, for instance, Elliott et al. [35]) is satisfied under $\mathbb{P}$ by the filtrations $\mathbb{F}$ and $\mathbb{G}$ so that, for every $u \in \mathbb{R}_+$,

$$\mathbb{P}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) = \mathbb{P}\left(\tau > u \,\middle|\, \mathcal{F}_u\right), \quad \forall\, t \in [u, \infty).$$

The main tool in our construction of a random time with a given Azéma supermartingale is a locally equivalent change of a probability measure. For this reason, we first present some results related to Girsanov's theorem in the present set-up.

Properties of $(\mathbb{P},\mathbb{G})$-Martingales

It order to define the Radon-Nikodým density process, we first analyze the properties of $(\mathbb{P},\mathbb{G})$-martingales. The following auxiliary result is based on El Karoui et al. [36], in the sense that it can be seen as a consequence of Theorem 5.7 therein. For the sake of completeness, we provide a simple proof of Proposition 3.3.1. In what follows, Z stands for a càdlàg, $\mathbb{F}$-adapted, $\mathbb{P}$-integrable process, whereas $Z_t(u)$ denotes an $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable map, where $\mathcal{O}(\mathbb{F})$ stands for the $\mathbb{F}$-optional σ-field in $\Omega\times\mathbb{R}_+$ (for details, see [36]).

Proposition 3.3.1. *Assume that the H-hypothesis is satisfied under $\mathbb{P}$ by the filtrations $\mathbb{F}$ and $\mathbb{G}$. Let the $\mathbb{G}$-adapted, $\mathbb{P}$-integrable process $Z^{\mathbb{G}}$ be given by the formula*

$$Z_t^{\mathbb{G}} = Z_t\mathbb{1}_{\{\tau>t\}} + Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}, \quad \forall\, t\in\mathbb{R}_+, \tag{3.11}$$

where:
(i) the projection of $Z^{\mathbb{G}}$ onto $\mathbb{F}$, which is defined by

$$Z_t^{\mathbb{F}} := \mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}}\,\middle|\,\mathcal{F}_t\right) = Z_t\,\mathbb{P}\left(\tau>t\,\middle|\,\mathcal{F}_t\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}\,\middle|\,\mathcal{F}_t\right),$$

is a $(\mathbb{P},\mathbb{F})$-martingale,
(ii) for any fixed $u\in\mathbb{R}_+$, the process $(Z_t(u),\, t\in[u,\infty))$ is a $(\mathbb{P},\mathbb{F})$-martingale.
Then the process $Z^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$-martingale.

Proof. Let us take $s<t$. Then

$$\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}}\,\middle|\,\mathcal{G}_s\right) = \mathbb{E}_{\mathbb{P}}\left(Z_t\mathbb{1}_{\{\tau>t\}}\,\middle|\,\mathcal{G}_s\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{s<\tau\le t\}}\,\middle|\,\mathcal{G}_s\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{G}_s\right) = I_1+I_2+I_3.$$

For I_1 and I_2, we apply the standard formula

$$I_1+I_2 = \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s^{\mathbb{P}}}\,\mathbb{E}_{\mathbb{P}}\left(Z_tG_t^{\mathbb{P}}\,\middle|\,\mathcal{F}_s\right) + \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s^{\mathbb{P}}}\,\mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{s<\tau\le t\}}\,\middle|\,\mathcal{F}_s\right),$$

whereas for I_3, we obtain

$$I_3 = \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{G}_s\right) = \mathbb{1}_{\{\tau\le s\}}\mathbb{E}_{\mathbb{P}}\left(Z_t(u)\,\middle|\,\mathcal{F}_s\right)_{u=\tau} = \mathbb{1}_{\{\tau\le s\}}\mathbb{E}_{\mathbb{P}}\left(Z_s(u)\,\middle|\,\mathcal{F}_s\right)_{u=\tau} = \mathbb{1}_{\{\tau\le s\}}Z_s(\tau),$$

where the first equality holds under the H-hypothesis[1] (see Section 3.2 in El Karoui et al. [36]) and the second follows from (ii). It thus suffices to show that $I_1+I_2 = Z_s\mathbb{1}_{\{\tau>s\}}$. Condition (i) yields

$$\mathbb{E}_{\mathbb{P}}\left(Z_tG_t^{\mathbb{P}}\,\middle|\,\mathcal{F}_s\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}\,\middle|\,\mathcal{F}_s\right) - \mathbb{E}_{\mathbb{P}}\left(Z_s(\tau)\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{F}_s\right) = Z_sG_s^{\mathbb{P}}.$$

Therefore,

$$I_1+I_2 = \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s^{\mathbb{P}}}\Big(Z_sG_s^{\mathbb{P}} + \mathbb{E}_{\mathbb{P}}\left((Z_s(\tau)-Z_t(\tau))\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{F}_s\right)\Big) = Z_s\mathbb{1}_{\{\tau>s\}},$$

where the last equality holds since

$$\mathbb{E}_{\mathbb{P}}\left((Z_s(\tau)-Z_t(\tau))\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{F}_s\right) = \mathbb{1}_{\{\tau\le s\}}\mathbb{E}_{\mathbb{P}}\left((Z_s(u)-Z_t(u))\,\middle|\,\mathcal{F}_s\right)_{u=\tau} = 0.$$

For the last equality in the formula above, we have again used condition (ii) in Proposition 3.3.1. □

In order to define a probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ under which (3.1) holds, we will search for a process $Z^{\mathbb{G}}$ satisfying the following set of assumptions.

[1] Essentially, this equality holds, since under the H-hypothesis the σ-fields $\mathcal{F}_t$ and $\mathcal{G}_s$ are conditionally independent given $\mathcal{F}_s$.

Assumption 3.3.1. The process $Z^{\mathbb{G}}$ is a $\mathbb{G}$-adapted and $\mathbb{P}$-integrable process given by

$$Z_t^{\mathbb{G}} = Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau)\mathbb{1}_{\{\tau \leq t\}}, \quad \forall\, t \in \mathbb{R}_+, \tag{3.12}$$

such that the following properties are valid:
(A.1) the projection of $Z^{\mathbb{G}}$ onto $\mathbb{F}$ is equal to one, that is, $Z_t^{\mathbb{F}} := \mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \,\middle|\, \mathcal{F}_t\right) = 1$ for every $t \in \mathbb{R}_+$,
(A.2) the process $Z^{\mathbb{G}}$ is a strictly positive $(\mathbb{P}, \mathbb{G})$-martingale.

Remarks 3.3.1. Since $\mathbb{P}(\tau > 0) = 1$ is clear that $Z_0^{\mathbb{G}} = Z_0 = 1$, so that $\mathbb{E}_{\mathbb{P}}(Z_t^{\mathbb{G}}) = 1$ for every $t \in \mathbb{R}_+$. We will later define a probability measure $\mathbb{Q}$ using the process $Z^{\mathbb{G}}$ as the Radon-Nikodým density. Then condition (A.1) will imply that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ equals $\mathbb{P}$ and, together with the equality $Z = N$, will give us control over the Azéma supermartingale of τ under $\mathbb{Q}$. Let us also note that assumption (A.1) implies that condition (i) of Proposition 3.3.1 is trivially satisfied.

The following lemma provides a simple condition, which is equivalent to property (A.1).

Lemma 3.3.1. *The projection of $Z^{\mathbb{G}}$ on $\mathbb{F}$ equals $Z_t^{\mathbb{F}} := \mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \,\middle|\, \mathcal{F}_t\right) = 1$ if and only if the processes Z and $Z_t(\tau)$ satisfy the following relationship*

$$Z_t = \frac{1 - \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau \leq t\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)}. \tag{3.13}$$

Proof. Straightforward calculations yield

$$\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau)\mathbb{1}_{\{\tau \leq t\}} \,\middle|\, \mathcal{F}_t\right) = Z_t\, \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau \leq t\}} \,\middle|\, \mathcal{F}_t\right) = 1.$$

The last equality is equivalent to formula (3.13). □

We find it convenient to work with the following assumption, which is more explicit and slightly stronger than Assumption 3.3.1.

Assumption 3.3.2. We postulate that the process Z and the map $Z_t(u)$ are such that:
(B.1) equality (3.13) is satisfied,
(B.2) for every $u \in \mathbb{R}_+$, the process $(Z_t(u),\, t \in [u, \infty))$ is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale.

Lemma 3.3.2. *Assumption 3.3.2 implies Assumption 3.3.1.*

Proof. In view of Lemma 3.3.1, the conditions (A.1) and (B.1) are equivalent. In view of Proposition 3.3.1, conditions (B.1) and (B.2) imply (A.2). □

Remarks 3.3.2. It is not true that Assumption 3.3.1 implies Assumption 3.3.2, since it is not true that Assumption 3.3.1 implies condition (B.2), in general. However, if the intensity $(\lambda_u, u \in \mathbb{R}_+)$ of τ under $\mathbb{P}$ exists then one can show that for any $u \in \mathbb{R}_+$ the process $(Z_t(u)\lambda_u G_u,\, t \in [u, \infty))$ is a $(\mathbb{P}, \mathbb{F})$-martingale (see El Karoui et al. [36]). This property implies in turn condition (B.2) provided that the intensity process λ does not vanish.

Girsanov's Theorem

To establish a suitable version of Girsanov's theorem, we need to specify a set-up in which the H-hypothesis is satisfied. Let Λ be an $\mathbb{F}$-adapted, continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. We define the random time τ using the *canonical construction*, that is, by setting

$$\tau = \inf\{t \in \mathbb{R}_+ : \Lambda_t \geq \Theta\}, \tag{3.14}$$

where Θ is an exponentially distributed random variable with parameter 1, independent of $\mathcal{F}_\infty$, and defined on a suitable extension of the space Ω. In fact, we will formally identify the probability space Ω with its extension, so that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ will be defined on the same space.

It is easy to check that in the case of the canonical construction, the H-hypothesis is satisfied under $\mathbb{P}$ by the filtrations $\mathbb{F}$ and $\mathbb{G}$. Moreover, the Azéma supermartingale of τ with respect to $\mathbb{F}$ under $\mathbb{P}$ equals

$$G_t^{\mathbb{P}} := \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+.$$

Under Assumption 3.3.2, the strictly positive $(\mathbb{P}, \mathbb{G})$-martingale $Z^{\mathbb{G}}$ given by (cf. (3.12))

$$Z_t^{\mathbb{G}} = Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau)\mathbb{1}_{\{\tau \le t\}}, \quad \forall\, t \in \mathbb{R}_+, \tag{3.15}$$

defines a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_\infty)$, locally equivalent to $\mathbb{P}$, by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\, \mathcal{G}_t = Z_t^{\mathbb{G}} \tag{3.16}$$

for every $t \in \mathbb{R}_+$.

The next result describes the conditional distributions of τ under a locally equivalent probability measure $\mathbb{Q}$.

Proposition 3.3.2. *Under Assumption 3.3.2, let the probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ be given by (3.15)–(3.16). Then the following properties hold:*
(i) the restriction of $\mathbb{Q}$ to the filtration $\mathbb{F}$ is equal to $\mathbb{P}$,
(ii) the Azéma supermartingale of τ under $\mathbb{Q}$ satisfies

$$G_t^{\mathbb{Q}} := \mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = Z_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+, \tag{3.17}$$

(iii) for every $u \in \mathbb{R}_+$,

$$\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(Z_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t\right), & t \le u, \\ Z_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u < \tau \le t\}} \,\middle|\, \mathcal{F}_t\right), & t \ge u. \end{cases} \tag{3.18}$$

Proof. Part (i) follows immediately from Assumption 3.3.2 and the fact that (B.1) is equivalent to (A.1). For (ii), it suffices to check that the second equality in (3.17) holds. By the abstract Bayes formula

$$\mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = \frac{\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \mathbb{1}_{\{\tau > t\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \,\middle|\, \mathcal{F}_t\right)} = \frac{\mathbb{E}_{\mathbb{P}}\left(Z_t \mathbb{1}_{\{\tau > t\}} \,\middle|\, \mathcal{F}_t\right)}{Z_t^{\mathbb{F}}} = Z_t\, \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = Z_t e^{-\Lambda_t},$$

as expected. Using the abstract Bayes formula, we can also find expressions for the conditional probabilities $\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right)$ for every $u, t \in \mathbb{R}_+$. For a fixed $u \in \mathbb{R}_+$ and every $t \in [0, u]$, we simply have that

$$\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) = \mathbb{Q}\left(\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_u\right) \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Z_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t\right).$$

For a fixed $u \in \mathbb{R}_+$ and every $t \in [u, \infty)$, we obtain

$$\begin{aligned} \mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) &= \frac{\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \mathbb{1}_{\{\tau > u\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \,\middle|\, \mathcal{F}_t\right)} = \mathbb{E}_{\mathbb{P}}\left(\left(Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau)\mathbb{1}_{\{\tau \le t\}}\right)\mathbb{1}_{\{\tau > u\}} \,\middle|\, \mathcal{F}_t\right) \\ &= \mathbb{E}_{\mathbb{P}}\left(Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau)\mathbb{1}_{\{u < \tau \le t\}} \,\middle|\, \mathcal{F}_t\right) = Z_t\, \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u < \tau \le t\}} \,\middle|\, \mathcal{F}_t\right) \\ &= Z_t G_t^{\mathbb{P}} + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u < \tau \le t\}} \,\middle|\, \mathcal{F}_t\right) = Z_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u < \tau \le t\}} \,\middle|\, \mathcal{F}_t\right). \end{aligned}$$

This completes the proof. □

Remarks 3.3.3. (i) It is worth stressing that the H-hypothesis does not hold under $\mathbb{Q}$. This property follows immediately from (3.17), since under the H-hypothesis the Azéma supermartingale is necessarily a decreasing process.
(ii) It is not claimed that τ is a finite random time under $\mathbb{Q}$. Indeed, this property holds if and only if

$$\lim_{t \to \infty} \mathbb{Q}(\tau > t) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{Q}}(Z_t e^{-\Lambda_t}) =: c = 0,$$

otherwise, we have that $\mathbb{Q}(\tau = \infty) = c$. Of course, if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ then necessarily $\mathbb{Q}(\tau < \infty) = 1$ since from (3.14) we deduce that $\mathbb{P}(\tau < \infty) = 1$ (this is a consequence of the assumption that $\Lambda_\infty = \infty$).
(iii) We observe that the formula

$$G_t^{\mathbb{Q}} := \mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = Z_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+, \tag{3.19}$$

represents the multiplicative decomposition of the Azéma supermartingale $G^{\mathbb{Q}}$ if and only if the process Z is a $(\mathbb{Q}, \mathbb{F})$-local martingale or, equivalently, if Z is a $(\mathbb{P}, \mathbb{F})$-local martingale (it is worth stressing that Assumption 3.3.2 does not imply that Z is a $(\mathbb{P}, \mathbb{F})$-martingale; see Example 3.3.1 below). In other words, an equivalent change of a probability measure may result in a change of the decreasing component in the multiplicative decomposition as well. The interested reader is referred to Section 6 in El Karoui et al. [36] for a more detailed analysis of the change of a probability measure in the framework of the so-called *density approach* to the modelling of a random time.

Example 3.3.1. To illustrate the above remark, let us set $Z_t(u) = 1/2$ for every $u \in \mathbb{R}_+$ and $t \in [0, u]$. Then the process $Z^{\mathbb{G}}$, which is given by the formula

$$Z_t^{\mathbb{G}} = \frac{1 - (1/2)\mathbb{P}\left(\tau \le t \,\middle|\, \mathcal{F}_t\right)}{\mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)} \mathbb{1}_{\{\tau > t\}} + (1/2)\mathbb{1}_{\{\tau \le t\}},$$

satisfies Assumption 3.3.2. The process Z is not a $(\mathbb{P}, \mathbb{F})$-local martingale, however, since

$$Z_t = \frac{1 + \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)}{2\mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)} = \frac{1 + e^{-\Lambda_t}}{2e^{-\Lambda_t}}.$$

Moreover, the Azéma supermartingale of τ under $\mathbb{Q}$ equals, for every $t \in \mathbb{R}_+$,

$$\mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = Z_t e^{-\Lambda_t} = \frac{1}{2}(1 + e^{-\Lambda_t}) = e^{-\widehat{\Lambda}_t},$$

where $\widehat{\Lambda}$ is an $\mathbb{F}$-adapted, continuous, increasing process different from Λ. Note, however, that $Z^{\mathbb{G}}$ is not a uniformly integrable $(\mathbb{P}, \mathbb{G})$-martingale and $\mathbb{Q}$ is not equivalent to $\mathbb{P}$ on $\mathcal{G}_\infty$ since $\widehat{\Lambda}_\infty = \ln 2 < \infty$, so that $\mathbb{Q}(\tau < \infty) < 1$. Note that the martingale $Z^{\mathbb{G}}$ has a jump at time τ and this in fact implies the processes Λ and $\widehat{\Lambda}$ do not coincide. We will see in the sequel (cf. Lemma 3.4.2) that, under mild technical assumptions, it is necessary to set $Z_t(t) = Z_t$ when solving Problem (P), so that the density process $Z^{\mathbb{G}}$ is continuous at τ. This is by no means surprising, since this equality was identified in El Karoui et al. [36] within the density approach as the crucial condition for the preservation of the $\mathbb{F}$-intensity of a random time τ under an equivalent change of a probability measure.

3.4 Construction Through a Change of Measure

We are in the position to address the issue of finding a solution $(\tau, \mathbb{Q})$ to Problem (P). Let N be a strictly positive $(\mathbb{P}, \mathbb{F})$-local martingale with $N_0 = 1$ and let Λ be an $\mathbb{F}$-adapted, continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. We postulate, in addition, that $G = Ne^{-\Lambda} \le 1$. Recall that a strictly positive local martingale is a supermartingale; this implies, in particular, that the process N is $\mathbb{P}$-integrable.

Before we proceed to an explicit construction of a random time τ and a probability measure $\mathbb{Q}$, let us show that for any solution $(\tau, \mathbb{Q})$ to Problem (P), we necessarily have that $\mathbb{Q}(\tau < \infty) = 1$.

Lemma 3.4.1. *For any solution $(\tau, \mathbb{Q})$ to Problem (P), we have that $\mathbb{Q}(\tau = \infty) = 0$.*

Proof. Note first that

$$\mathbb{Q}(\tau = \infty) = \lim_{t \to \infty} \mathbb{Q}(\tau > t) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{Q}}\left(\mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)\right) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{P}}\left(\mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)\right) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{P}}\left(N_t e^{-\Lambda_t}\right).$$

Since N is a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale, and thus a positive supermartingale, we have that $\lim_{t\to\infty} N_t =: N_\infty < \infty$, $\mathbb{P}$-a.s. By assumption $0 \le N_t e^{-\Lambda_t} \le 1$, and thus the dominated convergence theorem yields

$$\mathbb{Q}(\tau=\infty) = \lim_{t\to\infty} \mathbb{E}_{\mathbb{P}}\left(N_t e^{-\Lambda_t}\right) = \mathbb{E}_{\mathbb{P}}\left(\lim_{t\to\infty} N_t e^{-\Lambda_t}\right) = \mathbb{E}_{\mathbb{P}}\left(N_\infty \lim_{t\to\infty} e^{-\Lambda_t}\right) = 0,$$

where the last equality follows from the assumption that $\Lambda_\infty = \infty$. □

In the first step, using the canonical construction, we define a random time τ by formula (3.14). In the second step, we propose a suitable change of a probability measure.

In order to define a probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ under which (2.1) holds, we wish to employ Proposition 3.3.2 with $Z = N$. To this end, we postulate that Assumption 3.3.2 is satisfied by $Z = N$ and a judiciously selected $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$. The choice of a map $Z_t(u)$, for a given in advance process N, is studied in what follows.

Let $\mathbb{Q}$ be defined by (3.15) with Z replaced by N. Then, by Proposition 3.3.2, we conclude that the Azéma supermartingale of τ under $\mathbb{Q}$ equals

$$G^{\mathbb{Q}}_t := \mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = N_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{3.20}$$

By the same token, formula (3.18) remains valid when Z is replaced by N.

Our next goal is to investigate Assumption 3.3.2, which was crucial in the proof of Proposition 3.3.2 and thus also in obtaining equality (3.20). For this purpose, let us formulate the following auxiliary problem, which combines Assumption 3.3.2 with the assumption that $N = Z$.

Problem (P.1) Let a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale N with $N_0 = 1$ be given. Find an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$ such that the following conditions are satisfied:
(i) for every $t \in \mathbb{R}_+$,

$$1 - N_t\, \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau \le t\}} \,\middle|\, \mathcal{F}_t\right), \tag{3.21}$$

(ii) for any fixed $u \in \mathbb{R}_+$, the process $(Z_t(u),\, t \in [u,\infty))$ is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale.

Of course, equality (3.21) is obtained by combining (B.1) with the equality $Z = N$. If, for a given process N, we can find a solution $(Z_t(u),\, t \in [u,\infty))$ to Problem (P.1) then the pair $(Z_t, Z_t(u)) = (N_t, Z_t(u))$ will satisfy Assumption 3.3.2.

To examine the existence of a solution to Problem (P.1), we note first that formula (3.21) can be represented as follows

$$1 - G^{\mathbb{P}}_t N_t = \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau \le t\}} \,\middle|\, \mathcal{F}_t\right) = \int_0^t Z_t(u)\, d\mathbb{P}\left(\tau \le u \,\middle|\, \mathcal{F}_t\right).$$

Since the H-hypothesis is satisfied under $\mathbb{P}$, the last formula is equivalent to

$$1 - G^{\mathbb{P}}_t N_t = \int_0^t Z_t(u)\, d\mathbb{P}\left(\tau \le u \,\middle|\, \mathcal{F}_u\right) = \int_0^t Z_t(u)\, d\big(1 - G^{\mathbb{P}}_u\big) = -\int_0^t Z_t(u)\, dG^{\mathbb{P}}_u.$$

Since we work under the standing assumption that $G^{\mathbb{P}}_t = e^{-\Lambda_t}$, we thus obtain the following equation, which is equivalent to (3.21)

$$N_t e^{-\Lambda_t} = 1 + \int_0^t Z_t(u)\, de^{-\Lambda_u}. \tag{3.22}$$

We conclude that, within the present set-up, Problem (P.1) is equivalent to the following one.

Problem (P.2) Let a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale N with $N_0 = 1$ be given. Find an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$ such that the following conditions hold:
(i) for every $t \in \mathbb{R}_+$

$$N_t e^{-\Lambda_t} = 1 + \int_0^t Z_t(u)\, de^{-\Lambda_u}, \tag{3.23}$$

(ii) for any fixed $u \in \mathbb{R}_+$, the process $(Z_t(u),\, t \in [u, \infty))$ is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale.

Assume that $\widehat{Z}_t := Z_t(t)$ is an $\mathbb{F}$-optional process. We will now show that the equality $N = \widehat{Z}$ necessarily holds for any solution to (3.23), in the sense made precise in Lemma 3.4.2. In particular, it follows immediately from this result that the processes N and $\widehat{Z}$ are indistinguishable when $\widehat{Z}$ is an $\mathbb{F}$-adapted, càdlàg process and the process Λ has strictly increasing sample paths (for instance, when $\Lambda_t = \int_0^t \lambda_u \, du$ for some strictly positive *intensity process* λ).

Lemma 3.4.2. *Suppose that $Z_t(u)$ solves Problem (P.2) and the process $(\widehat{Z}_t,\, t \in \mathbb{R}_+)$ given by $\widehat{Z}_t = Z_t(t)$ is $\mathbb{F}$-optional. Then $N = \widehat{Z}$, ν-a.e., where the measure ν on $(\Omega \times \mathbb{R}_+, \mathcal{O}(\mathbb{F}))$ is generated by the increasing process Λ, that is, for every $s < t$ and any bounded, $\mathbb{F}$-optional process V*

$$\nu(\mathbb{1}_{[s,t[}V) = \mathbb{E}_{\mathbb{P}}\Big(\int_s^t V_u \, d\Lambda_u \Big).$$

Proof. The left-hand side in (3.23) has the following Doob-Meyer decomposition

$$N_t e^{-\Lambda_t} = 1 + \int_0^t e^{-\Lambda_u} \, dN_u + \int_0^t N_u \, de^{-\Lambda_u}, \tag{3.24}$$

whereas the right-hand side in (3.23) can be represented as follows

$$1 + \int_0^t Z_t(u) \, de^{-\Lambda_u} = 1 + \int_0^t (Z_t(u) - \widehat{Z}_u) \, de^{-\Lambda_u} + \int_0^t \widehat{Z}_u \, de^{-\Lambda_u} = 1 + I_1(t) + I_2(t), \tag{3.25}$$

where I_2 is an $\mathbb{F}$-adapted, continuous process of finite variation. We will show that I_1 is a $(\mathbb{P}, \mathbb{F})$-martingale, so that right-hand side in (3.25) yields the Doob-Meyer decomposition as well. To this end, we need to show that the equality $\mathbb{E}_{\mathbb{P}}\left(I_1(t) \,\middle|\, \mathcal{F}_s\right) = I_1(s)$ holds for every $s < t$ or, equivalently,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^t Z_t(u) \, de^{-\Lambda_u} - \int_0^s Z_s(u) \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \widehat{Z}_u \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right). \tag{3.26}$$

We first observe that, for every $s < t$,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^s Z_s(u) \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^s \mathbb{E}_{\mathbb{P}}\left(Z_t(u) \,\middle|\, \mathcal{F}_s \right) \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^s Z_t(u) \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right),$$

and thus the right-hand side in (3.26) satisfies

$$\mathbb{E}_{\mathbb{P}}\left(\int_s^t Z_t(u) \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \mathbb{E}_{\mathbb{P}}\left(Z_t(u) \,\middle|\, \mathcal{F}_u \right) \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \widehat{Z}_u \, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s \right),$$

where we have used the following equality, which holds for every $\mathbb{F}$-adapted, continuous process A of finite variation and every càdlàg process V (not necessarily $\mathbb{F}$-adapted)

$$\mathbb{E}_{\mathbb{P}}\left(\int_s^t V_u \, dA_u \,\middle|\, \mathcal{F}_s \right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \mathbb{E}_{\mathbb{P}}\left(V_u \,\middle|\, \mathcal{F}_u \right) \, dA_u \,\middle|\, \mathcal{F}_s \right).$$

We thus see that (3.26) holds, so that I_1 is a $(\mathbb{P}, \mathbb{F})$-martingale. By comparing the right-hand sides in (3.24) and (3.25) and using the uniqueness of the Doob-Meyer decomposition, we conclude that

$$\int_0^t (N_u - \widehat{Z}_u) \, de^{-\Lambda_u} = 0, \quad \forall \, t \in \mathbb{R}_+.$$

The formula above implies that $N = \widehat{Z}$, $\widetilde{\nu}$-a.e., where the measure $\widetilde{\nu}$ on $(\Omega \times \mathbb{R}_+, \mathcal{O}(\mathbb{F}))$ is generated by the decreasing process $e^{-\Lambda_u}$. It is easily seen that the measures $\widetilde{\nu}$ and ν are equivalent and thus $N = \widehat{Z}$, ν-a.e. $\square$

To address the issue of existence of a solution to Problem (P.2) (note that it is not claimed that a solution $Z_t(u)$ to Problem (P.2) is unique), we start by postulating that, as in the filtering case described in Section 3.2, an $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$ satisfies: $Z_t(u) = X_t Y_{t\wedge u}$ for some $\mathbb{F}$-adapted, continuous, strictly positive processes X and Y. It is then easy to check that condition (ii) implies that the process X is necessarily a $(\mathbb{P},\mathbb{F})$-martingale. Moreover, equation (3.23) becomes

$$N_t e^{-\Lambda_t} = 1 + X_t \int_0^t Y_u \, de^{-\Lambda_u}. \tag{3.27}$$

Note that at this stage we are searching for a pair (X,Y) of strictly positive, $\mathbb{F}$-adapted processes such that X is a $(\mathbb{P},\mathbb{F})$-martingale and equality (3.27) holds for every $t \in \mathbb{R}_+$. In view of Lemma 3.4.2, it is also natural to postulate that $N = XY$. We will then be able to find a simple relation between processes X and Y (see formula (3.28) below) and thus to reduce the dimensionality of the problem.

Lemma 3.4.3. *(i) Assume that a pair (X,Y) of strictly positive processes is such that the process $Z_t(u) = X_t Y_{t\wedge u}$ solves Problem (P.2) and the equality $N = XY$ holds. Then the process X is a $(\mathbb{P},\mathbb{F})$-martingale and the process Y equals*

$$Y_t = Y_0 + \int_0^t e^{\Lambda_u} \, d\left(\frac{1}{X_u}\right). \tag{3.28}$$

(ii) Conversely, if X is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale and the process Y given by (3.28) is strictly positive, then the process $Z_t(u) = X_t Y_{t\wedge u}$ solves Problem (P.2) for the process $N = XY$.

Proof. For part (i), we observe that under the assumption that $N = XY$, equation (3.27) reduces to

$$X_t Y_t e^{-\Lambda_t} = 1 + X_t \int_0^t Y_u \, de^{-\Lambda_u}, \tag{3.29}$$

which in turn is equivalent to

$$Y_t e^{-\Lambda_t} = \frac{1}{X_t} + \int_0^t Y_u \, de^{-\Lambda_u}. \tag{3.30}$$

The integration by parts formula yields

$$\frac{1}{X_t} = Y_0 + \int_0^t e^{-\Lambda_u} \, dY_u$$

and this in turn is equivalent to (3.28). To establish part (ii), we first note that (3.28) implies (3.29), which means that (3.23) is satisfied by the processes $N = XY$ and $Z_t(u) = X_t Y_{t\wedge u}$. It is also clear that for any fixed $u \in \mathbb{R}_+$, the process $(Z_t(u),\, t \in [u,\infty))$ is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale. Let us also note that, by the Itô formula, the process XY satisfies

$$d(X_t Y_t) = Y_t \, dX_t - e^{\Lambda_t}(1/X_t) \, dX_t, \tag{3.31}$$

and thus it is a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale. □

We conclude that in order to find a solution $Z_t(u) = X_t Y_{t\wedge u}$ to Problem (P.2), it suffices to solve the following problem.

Problem (P.3) Assume that we are given a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale N with $N_0 = 1$ and an $\mathbb{F}$-adapted, continuous, increasing process Λ with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. Find a strictly positive $(\mathbb{P},\mathbb{F})$-martingale X such that for the process Y given by (3.28) we have that $N = XY$.

The following corollary is an easy consequence of part (ii) in Lemma 3.4.3.

Corollary 3.4.1. *Assume that a process X solves Problem (P.3) and let Y be given by (3.28). Then the processes $Z = N$ and $Z_t(u) = X_t Y_{t\wedge u}$ solve Problem (P.2) and thus they satisfy Assumption 3.3.2.*

3.5 Case of a Brownian Filtration

The aim of this section is to examine the existence of a solution to Problem (P.3) under the following standing assumptions:
(i) the filtration $\mathbb{F}$ is generated by a Brownian motion W,
(ii) we are given an $\mathbb{F}$-adapted, continuous, increasing process Λ with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$ and a strictly positive $(\mathbb{P}, \mathbb{F})$-local martingale N satisfying

$$N_t = 1 + \int_0^t \nu_u N_u \, dW_u, \quad \forall \, t \in \mathbb{R}_+, \tag{3.32}$$

for some $\mathbb{F}$-predictable process ν,
(iii) the inequality $G_t := N_t e^{-\Lambda_t} < 1$ holds for every $t > 0$, so that $N_t < e^{\Lambda_t}$ for every $t > 0$.

We start by noting that X is postulated to be a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale and thus it is necessarily given by

$$X_t = \exp\left(\int_0^t x_s \, dW_s - \frac{1}{2} \int_0^t x_s^2 \, ds \right), \quad \forall \, t \in \mathbb{R}_+, \tag{3.33}$$

where the process x is yet unknown. The goal is to specify x in terms of N and Λ in such a way that the equality $N = XY$ will hold for Y given by (3.28).

Lemma 3.5.1. *Let X be given by (3.33) with the process x satisfying*

$$x_t = \frac{\nu_t N_t}{N_t - e^{\Lambda_t}}, \quad \forall \, t \in \mathbb{R}_+. \tag{3.34}$$

Assume that x is a square-integrable process. Then the equality $N = XY$ holds, where the process Y given by (3.28) with $Y_0 = 1$.

Proof. Using (3.31) and (3.33), we obtain

$$d(X_t Y_t) = Y_t \, dX_t - e^{\Lambda_t} (1/X_t) \, dX_t = Y_t x_t X_t \, dW_t - e^{\Lambda_t} (1/X_t) \, x_t X_t \, dW_t,$$

and thus

$$d(X_t Y_t) = x_t \big(X_t Y_t - e^{\Lambda_t} \big) \, dW_t. \tag{3.35}$$

Let us denote $V = XY$. Then V satisfies the following SDE

$$dV_t = x_t \, (V_t - e^{\Lambda_t}) \, dW_t. \tag{3.36}$$

In view of (3.34), it is clear that the process N solves this equation as well. Hence to show that the equality $N = XY$ holds, it suffices to show that a solution to the SDE (3.36) is unique. We note that we deal here with the integral equation of the form

$$V_t = H_t + \int_0^t x_u V_u \, dW_u. \tag{3.37}$$

We will show that a solution to (3.37) is unique. To this end, we argue by contradiction. Suppose that V^i, $i = 1, 2$ are any two solutions to (3.37). Then the process $U = V^1 - V^2$ satisfies

$$dU_t = x_t U_t \, dW_t, \quad U_0 = 0, \tag{3.38}$$

which admits the obvious solution $U = 0$. Suppose that $\widehat{U}$ is a non-null solution to (3.38). Then the Doléans-Dade equation $dX_t = x_t X_t \, dW_t$, $X_0 = 1$ would admit the usual solution X given by (3.33) and another solution $X + \widehat{U} \neq X$, and this is well known to be false. We conclude that (3.38) admits a unique solution, and this in turn implies the uniqueness of a solution to (3.37). This shows that $N = XY$, as was stated. $\square$

The following result is an immediate consequence of Corollary 3.4.1 and Lemma 3.5.1.

Corollary 3.5.1. *Let the filtration $\mathbb{F}$ be generated by a Brownian motion W on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$. Assume that we are given an $\mathbb{F}$-adapted, continuous supermartingale G such that $0 < G_t < 1$ for every $t > 0$ and*

$$G_t = N_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+, \tag{3.39}$$

where Λ is an $\mathbb{F}$-adapted, continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$, and N is a strictly positive $\mathbb{F}$-local martingale, so that there exists an $\mathbb{F}$-predictable process ν such that

$$N_t = 1 + \int_0^t \nu_u N_u \, dW_u, \quad \forall\, t \in \mathbb{R}_+. \tag{3.40}$$

Let X be given by (3.33) with the process x satisfying

$$x_t = \frac{\nu_t G_t}{G_t - 1}, \quad \forall\, t \in \mathbb{R}_+. \tag{3.41}$$

Then:
(i) the equality $N = XY$ holds for the process Y given by (3.28),
(ii) the processes $Z = N$ and $Z_t(u) = X_t Y_{t\wedge u}$ satisfy Assumption 3.3.2.

In the next result, we denote by τ the random time defined by the canonical construction on a (possibly extended) probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$. By construction, the Azéma supermartingale of τ with respect to $\mathbb{F}$ under $\mathbb{P}$ equals

$$G_t^{\mathbb{P}} := \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+.$$

Let us note that, since the H-hypothesis is satisfied, the Brownian motion W remains a Brownian motion with respect to the enlarged filtration $\mathbb{G}$ under $\mathbb{P}$. It is still a Brownian motion under $\mathbb{Q}$ with respect to the filtration $\mathbb{F}$, since the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$. However, the process W is not necessarily a Brownian motion under $\mathbb{Q}$ with respect to the enlarged filtration $\mathbb{G}$.

The following result furnishes a solution to Problem (P) within the set-up described at the beginning of this section.

Proposition 3.5.1. *Under the assumptions of Corollary 3.5.1, we define a probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ by the Radon-Nikodým density process $Z^{\mathbb{G}}$ given by formula (3.12) with $Z_t = X_t Y_t = N_t$ and $Z_t(u) = X_t Y_{t\wedge u}$ or, more explicitly,*

$$Z_t^{\mathbb{G}} = N_t \mathbb{1}_{\{\tau > t\}} + X_t Y_{t\wedge\tau} \mathbb{1}_{\{\tau \le t\}}, \quad \forall\, t \in \mathbb{R}_+. \tag{3.42}$$

Then the Azéma supermartingale of τ with respect to $\mathbb{F}$ under $\mathbb{Q}$ satisfies

$$\mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = X_t Y_t e^{-\Lambda_t} = N_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{3.43}$$

Moreover, the conditional distribution of τ given $\mathcal{F}_t$ satisfies

$$\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(N_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t\right), & t < u, \\ N_t e^{-\Lambda_t} + X_t\, \mathbb{E}_{\mathbb{P}}\left(Y_\tau \mathbb{1}_{\{u < \tau \le t\}} \,\middle|\, \mathcal{F}_t\right), & t \ge u. \end{cases}$$

Proof. In view of Corollary 3.5.1, Assumption 3.3.2 is satisfied and thus the probability measure $\mathbb{Q}$ is well defined by the Radon-Nikodým density process $Z^{\mathbb{G}}$ given by (3.15), which is now equivalent to (3.42). Therefore, equality (3.43) is an immediate consequence of Proposition 3.3.2. Using (3.18), for every $u \in \mathbb{R}_+$, we obtain, for every $t \in [0, u]$

$$\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Z_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(N_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t\right),$$

whereas for every $t \in [u,\infty)$, we get

$$\begin{aligned}\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) &= Z_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right)\\ &= X_t Y_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(X_t Y_{t\wedge\tau}\mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right)\\ &= N_t e^{-\Lambda_t} + X_t\,\mathbb{E}_{\mathbb{P}}\left(Y_\tau\mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right),\end{aligned}$$

as required. □

Example 3.5.1. This example is related to the filtering problem examined in Section 3.2. Let $W = (W_t,\, t \in \mathbb{R}_+)$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and let $\mathbb{F}$ be its natural filtration. We wish to model a random time with the Azéma semimartingale with respect to the filtration $\mathbb{F}$ given by the solution to the following SDE (cf. (3.3))

$$dG_t = -\lambda G_t\,dt + \frac{b}{\sigma}\,G_t(1-G_t)\,dW_t, \quad G_0 = 1. \tag{3.44}$$

A comparison theorem for SDEs implies that $0 < G_t < 1$ for every $t > 0$. Moreover, by an application of the Itô formula, we obtain

$$G_t = N_t e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+,$$

where the martingale N satisfies

$$dN_t = \frac{b}{\sigma}(1-G_t)N_t\,dW_t. \tag{3.45}$$

As in Section 3.3, we start the construction of τ by first defining a random variable Θ with exponential distribution with parameter 1 and independent of $\mathcal{F}_\infty$ under $\mathbb{P}$ and by setting

$$\tau = \inf\{t \in \mathbb{R}_+ :\, \lambda t \ge \Theta\}.$$

In the second step, we propose an equivalent change of a probability measure. For this purpose, we note that the process x is here given by (cf. (3.34))

$$x_t = \frac{\nu_t N_t}{N_t - e^{\lambda t}} = -\frac{b}{\sigma}G_t,$$

and thus x is a bounded process. Next, in view of (3.33) and (3.45), the process X solves the SDE

$$dX_t = -\frac{b}{\sigma}G_t X_t\,dW_t,$$

and the process Y satisfies (cf. (3.28))

$$dY_t = e^{\lambda t}\,d\Big(\frac{1}{X_t}\Big) = \frac{N_t}{X_t}\Big(\frac{b}{\sigma}\,dW_t + \frac{b^2}{\sigma^2}\,G_t\,dt\Big).$$

The integration by parts formula yields

$$\begin{aligned}d\left(\frac{N_t}{X_t}\right) &= \frac{1}{X_t}\,dN_t + N_t\,d\left(\frac{1}{X_t}\right) + d\left[\frac{1}{X}, N\right]_t\\ &= \frac{N_t}{X_t}\frac{b}{\sigma}\,(1-G_t)\,dW_t + \frac{N_t}{X_t}\left(\frac{b}{\sigma}\,G_t\,dW_t + \frac{b^2}{\sigma^2}\,G_t^2\,dt\right) + \frac{N_t}{X_t}\frac{b^2}{\sigma^2}\,G_t(1-G_t)\,dt\\ &= \frac{N_t}{X_t}\Big(\frac{b}{\sigma}\,dW_t + \frac{b^2}{\sigma^2}\,G_t\,dt\Big).\end{aligned}$$

It is now easy to conclude that $N = XY$, as was expected. Under the probability measure $\mathbb{Q}$ introduced in Proposition 3.5.1, we have that

$$G_t^{\mathbb{Q}} := \mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = G_t = N_t e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+.$$

Moreover, the conditional distribution of τ given $\mathcal{F}_t$ satisfies

$$\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(N_u e^{-\lambda u} \,\middle|\, \mathcal{F}_t\right) = N_t e^{-\lambda u}, & t < u, \\ N_t e^{-\lambda t} + X_t\, \mathbb{E}_{\mathbb{P}}\left(Y_\tau \mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right), & t \ge u. \end{cases}$$

Since τ is here independent of $\mathcal{F}_\infty$ under $\mathbb{P}$, we obtain, for $t \ge u$,

$$\mathbb{E}_{\mathbb{P}}\left(Y_\tau \mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right) = \int_u^t Y_v \lambda e^{-\lambda v}\, dv = -\int_u^t Y_v\, de^{-\lambda v} = Y_u e^{-\lambda u} - Y_t e^{-\lambda t} + \left(\frac{1}{X_t} - \frac{1}{X_u}\right),$$

where the last equality can be deduced, for instance, from (3.30). Therefore, for every $t \ge u$

$$\mathbb{Q}\left(\tau > u \,\middle|\, \mathcal{F}_t\right) = 1 - \frac{X_t}{X_u} + X_t Y_u e^{-\lambda u}.$$

We conclude that, for every $t, u \in \mathbb{R}_+$,

$$\mathbb{Q}(\tau > u \,|\, \mathcal{F}_t) = 1 - \frac{X_t}{X_{u\wedge t}} + X_t Y_{u\wedge t} e^{-\lambda u}.$$

It is interesting to note that this equality agrees with the formula (3.7), which was established in Proposition 3.2.1 in the context of filtering problem using a different technique.

3.6 Applications to Valuation of Credit Derivatives

We will examine very succinctly the importance of the multiplicative decomposition of the Azéma supermartingale (i.e., the survival process) of a random time for the risk-neutral valuation of credit derivatives. Unless explicitly stated otherwise, we assume that the interest rate is null. This assumption is made for simplicity of presentation and, obviously, it can be easily relaxed.

As a risk-neutral probability, we will select either the probability measure $\mathbb{P}$ or the equivalent probability measure $\mathbb{Q}$ defined here on $(\Omega, \mathcal{G}_T)$, where T stands for the maturity date of a credit derivative. Recall that within the framework considered in this chapter the survival process of a random time τ is given under $\mathbb{P}$ and $\mathbb{Q}$ by the following formulae

$$G_t^{\mathbb{P}} := \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = e^{-\Lambda_t},$$

and

$$G_t^{\mathbb{Q}} := \mathbb{Q}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = N_t e^{-\Lambda_t},$$

respectively. The random time τ is here interpreted as the *default time* of a reference entity of a credit derivative.

We assume from now on that the increasing process Λ satisfies $\Lambda_t = \int_0^t \lambda_u\, du$ for some non-negative, $\mathbb{F}$-progressively measurable process λ. Then we have the following well known result.

Lemma 3.6.1. *The process M, given by the formula*

$$M_t = \mathbb{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \lambda_u\, du,$$

is a $\mathbb{G}$-martingale under the probability measures $\mathbb{P}$ and $\mathbb{Q}$.

The property established in Lemma 3.6.1 is frequently adopted in the financial literature as the definition of the *default intensity* λ. In the present set-up, Lemma 3.6.1 implies that the default intensity is the same under the equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$, despite the fact that the corresponding survival processes $G^{\mathbb{P}}$ and $G^{\mathbb{Q}}$ are different (recall that we postulate that N is a non-trivial local martingale). Hence the following question arises: is the specification of the default intensity λ sufficient for the risk-neutral valuation of credit derivatives related to a reference entity? Similarly as in El Karoui et al. [36], we will argue that the answer to this question is negative. To support this claim, we will show that the risk-neutral valuation of credit derivatives requires the full knowledge of the survival process, and thus the knowledge of the decreasing component Λ of the survival process in not sufficient for this purpose, in general.

To illustrate the importance of martingale component N for the valuation of credit derivatives, we first suppose $\Lambda = (\Lambda_t,\, t \in \mathbb{R}_+)$ is deterministic. We will argue that in that setting the process N has no influence on the prices of some simple credit derivatives, such as: a defaultable zero-coupon bond with zero recovery or a stylized credit default swap (CDS) with a deterministic protection payment. This means that the model calibration based on these assets will only allow us to recover the function Λ, but will provide no information regarding the local martingale component N of the survival process $G^{\mathbb{Q}}$. However, if the assumptions of the deterministic character of Λ and/or protection payment of a CDS are relaxed, then the corresponding prices will depend on the choice of N as well, and thus the explicit knowledge of N becomes important (for an example, see Corollary 3.6.1). As expected, this feature becomes even more important when we deal with a credit risk model in which the default intensity λ is stochastic, as is typically assumed in the financial literature.

3.6.1 Defaultable Zero-Coupon Bonds

By definition, the risk-neutral price under $\mathbb{Q}$ of the T-maturity defaultable zero-coupon bond with zero recovery equals, for every $t \in [0,T]$,

$$D^{\mathbb{Q}}(t,T) := \mathbb{Q}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{Q}(\tau > T \,|\, \mathcal{F}_t)}{G_t^{\mathbb{Q}}} = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{Q}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)}{G_t^{\mathbb{Q}}}. \tag{3.46}$$

Assuming that Λ is deterministic, we obtain the pricing formulae independent of N. Indeed, the risk-neutral price of the bond under $\mathbb{P}$ equals, for every $t \in [0,T]$,

$$D^{\mathbb{P}}(t,T) := \mathbb{P}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\, e^{-(\Lambda_T - \Lambda_t)}.$$

On the other hand, using (3.46) and the fact that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$, we obtain

$$\begin{aligned} D^{\mathbb{Q}}(t,T) &:= \mathbb{Q}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{1}{N_t e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{Q}}(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t) \\ &= \mathbb{1}_{\{\tau > t\}} \frac{1}{N_t e^{-\Lambda_t}}\, e^{-\Lambda_T}\, \mathbb{E}_{\mathbb{P}}(N_T \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau > t\}}\, e^{-(\Lambda_T - \Lambda_t)}, \end{aligned}$$

where we have assumed that N is a (true) $(\mathbb{P},\mathbb{F})$-martingale. If N is a strict local martingale then it is a strict supermartingale and thus $D^{\mathbb{Q}}(t,T) \leq D^{\mathbb{P}}(t,T)$.

If we allow for a stochastic process Λ, then the role of N in the valuation of defaultable zero-coupon bonds becomes important, as can be seen from the following expressions

$$D^{\mathbb{P}}(t,T) = \mathbb{1}_{\{\tau > t\}} \frac{1}{e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{P}}\big(e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)$$

and

$$D^{\mathbb{Q}}(t,T) = \mathbb{1}_{\{\tau > t\}} \frac{1}{N_t e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{Q}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big) = \mathbb{1}_{\{\tau > t\}} \frac{1}{N_t e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{P}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big),$$

where the last equality follows from the standing assumption that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$. We thus see that the inequality $D^{\mathbb{P}}(t,T) \neq D^{\mathbb{Q}}(t,T)$ is likely to hold when Λ is stochastic.

In the case of a stochastic interest rate, the bond valuation problem is more difficult. Let the discount factor $\beta = (\beta_t, t \in \mathbb{R}_+)$ be defined by

$$\beta_t = \exp\left(-\int_0^t r_s\, ds\right),$$

where the short-term interest rate process $r = (r_t,\, t \in \mathbb{R}_+)$ is assumed to be $\mathbb{F}$-adapted. Then the risk-neutral prices under $\mathbb{P}$ and $\mathbb{Q}$ of the T-maturity defaultable zero-coupon bond with zero recovery are given by the following expressions

$$D^{\mathbb{P}}(t,T) = \mathbb{1}_{\{\tau>t\}} \frac{1}{e^{-\Lambda_t}\beta_t}\, \mathbb{E}_{\mathbb{P}}\big(e^{-\Lambda_T}\beta_T \,|\, \mathcal{F}_t\big)$$

and

$$D^{\mathbb{Q}}(t,T) = \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}\beta_t}\, \mathbb{E}_{\mathbb{P}}\big(N_T e^{-\Lambda_T}\beta_T \,|\, \mathcal{F}_t\big).$$

Once again, it is clear that the role of the martingale component of the survival process is non-trivial, even in the case when the default intensity is assumed to be deterministic.

3.6.2 Credit Default Swaps

Let us now consider a stylized CDS with the protection payment process R and fixed spread κ, which gives protection over the period $[0,T]$. It is known that the risk-neutral price under $\mathbb{P}$ of this contract is given by the formula, for every $t \in [0,T]$,

$$S_t^{\mathbb{P}} := \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{t<\tau\le T\}} R_\tau - \kappa\big((T\wedge\tau) - (t\vee\tau)\big)\,\Big|\, \mathcal{G}_t\Big). \tag{3.47}$$

In the case where $\Lambda_t = \int_0^t \lambda_u\, du$, one can also show that (see Bielecki et al. [13])

$$S_t^{\mathbb{P}} = \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t^{\mathbb{P}}}\, \mathbb{E}_{\mathbb{P}}\bigg(\int_t^T G_u^{\mathbb{P}}(R_u\lambda_u - \kappa)\, du \,\bigg|\, \mathcal{F}_t\bigg). \tag{3.48}$$

Analogous formulae are valid under $\mathbb{Q}$, if we decide to choose $\mathbb{Q}$ as a risk-neutral probability.

To analyze the impact of N on the value of the CDS, let us first consider the special case when the default intensity λ and the protection payment R are assumed to be deterministic. In that case, the risk-neutral price of the CDS under $\mathbb{P}$ can be represented as follows

$$S_t^{\mathbb{P}} = \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t} \int_t^T e^{-\Lambda_u}(R_u\lambda_u - \kappa)\, du.$$

For the risk-neutral price under $\mathbb{Q}$, we obtain

$$\begin{aligned}
S_t^{\mathbb{Q}} &= \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t^{\mathbb{Q}}}\, \mathbb{E}_{\mathbb{Q}}\bigg(\int_t^T G_u^{\mathbb{Q}}(R_u\lambda_u - \kappa)\, du \,\bigg|\, \mathcal{F}_t\bigg) \\
&= \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t^{\mathbb{Q}}} \int_t^T \mathbb{E}_{\mathbb{P}}\big(G_u^{\mathbb{Q}} \,|\, \mathcal{F}_t\big)(R_u\lambda_u - \kappa)\, du \\
&= \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}} \int_t^T \mathbb{E}_{\mathbb{P}}\big(N_u \,|\, \mathcal{F}_t\big) e^{-\Lambda_u}(R_u\lambda_u - \kappa)\, du \\
&= \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t} \int_t^T e^{-\Lambda_u}(R_u\lambda_u - \kappa)\, du,
\end{aligned}$$

where in the second equality we have used once again the standing assumption that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$ and the last one holds provided that N is a $(\mathbb{P},\mathbb{F})$-martingale. It is thus clear that the equality $S_t^{\mathbb{P}} = S_t^{\mathbb{Q}}$ holds for every $t \in [0,T]$, so that the price of the CDS does not depend on the particular choice of N when λ and R are deterministic.

Let us now consider the case when the intensity λ is assumed to be deterministic, but the protection payment R is allowed to be stochastic. Since our goal is to provide an explicit example, we postulate that $R = 1/N$, though we do not pretend that this is a natural choice of the protection payment process R.

Corollary 3.6.1. *Let us set $R = 1/N$ and let us assume that the default intensity λ is deterministic. If the process N is a $(\mathbb{P}, \mathbb{F})$-martingale then the fair spread $\kappa_0^{\mathbb{P}}$ of the CDS under $\mathbb{P}$ equals*

$$\kappa_0^{\mathbb{P}} = \frac{\int_0^T \mathbb{E}_{\mathbb{P}}\big((N_u)^{-1}\big)\lambda_u e^{-\Lambda_u}\,du}{\int_0^T e^{-\Lambda_u}\,du} \tag{3.49}$$

and the fair spread $\kappa_0^{\mathbb{Q}}$ under $\mathbb{Q}$ satisfies

$$\kappa_0^{\mathbb{Q}} = \frac{1 - e^{-\Lambda_T}}{\int_0^T e^{-\Lambda_u}\,du} < \kappa_0^{\mathbb{P}}. \tag{3.50}$$

Proof. Recall that the fair spread at time 0 is defined as the level of κ for which the value of the CDS at time 0 equals zero, that is, $S_0(\kappa) = 0$. By applying formula (3.48) with $t = 0$ to risk-neutral probabilities $\mathbb{P}$ and $\mathbb{Q}$, we thus obtain

$$\kappa_0^{\mathbb{P}} = \frac{\mathbb{E}_{\mathbb{P}}\left(\int_0^T G_u^{\mathbb{P}} R_u \lambda_u\,du\right)}{\mathbb{E}_{\mathbb{P}}\left(\int_0^T G_u^{\mathbb{P}}\,du\right)}$$

and

$$\kappa_0^{\mathbb{Q}} = \frac{\mathbb{E}_{\mathbb{Q}}\left(\int_0^T G_u^{\mathbb{Q}} R_u \lambda_u\,du\right)}{\mathbb{E}_{\mathbb{Q}}\left(\int_0^T G_u^{\mathbb{Q}}\,du\right)}.$$

Let us first observe that the denominators in the two formulae above are in fact equal since

$$\mathbb{E}_{\mathbb{P}}\bigg(\int_0^T G_u^{\mathbb{P}}\,du\bigg) = \int_0^T e^{-\Lambda_u}\,du$$

and

$$\mathbb{E}_{\mathbb{Q}}\bigg(\int_0^T G_u^{\mathbb{Q}}\,du\bigg) = \mathbb{E}_{\mathbb{Q}}\bigg(\int_0^T N_t e^{-\Lambda_u}\,du\bigg) = \int_0^T \mathbb{E}_{\mathbb{Q}}(N_t) e^{-\Lambda_u}\,du = \int_0^T \mathbb{E}_{\mathbb{P}}(N_t) e^{-\Lambda_u}\,du = \int_0^T e^{-\Lambda_u}\,du$$

since the restriction of $\mathbb{Q}$ to $\mathbb{F}$ equals $\mathbb{P}$ and N is a $(\mathbb{P}, \mathbb{F})$-martingale, so that $\mathbb{E}_{\mathbb{P}}(N_t) = 1$ for every $t \in [0, T]$. For the numerators, using the postulated equality $R = 1/N$, we obtain

$$\mathbb{E}_{\mathbb{P}}\bigg(\int_0^T G_u^{\mathbb{P}} R_u \lambda_u\,du\bigg) = \mathbb{E}_{\mathbb{P}}\bigg(\int_0^T (N_u)^{-1} \lambda_u e^{-\Lambda_u}\,du\bigg) = \int_0^T \mathbb{E}_{\mathbb{P}}\big((N_u)^{-1}\big)\lambda_u e^{-\Lambda_u}\,du$$

and

$$\mathbb{E}_{\mathbb{Q}}\bigg(\int_0^T G_u^{\mathbb{Q}} R_u \lambda_u\,du\bigg) = \int_0^T \lambda_u e^{-\Lambda_u}\,du = 1 - e^{-\Lambda_T}.$$

This proves equalities in (3.49) and (3.50). The inequality in formula (3.50) follows from the fact that $1/N$ is a strict submartingale and thus $\mathbb{E}_{\mathbb{P}}((N_t)^{-1}) > 1$ for every $t \in \mathbb{R}_+$. $\square$

Formulae (3.49) and (3.50) make it clear that the fair spreads $\kappa_0^{\mathbb{P}}$ and $\kappa_0^{\mathbb{Q}}$ are not equal, in general. This example supports our claim that the knowledge of the default intensity λ, even in the case when λ is deterministic, is not sufficient for the determination of the risk-neutral price of a credit derivative, in general. To conclude, a specific way in which the default time of the underlying entity is modeled should always be scrutinized in detail, and, when feasible, the multiplicative decomposition of the associated survival process should be computed explicitly.

Chapter 4

Random Times and Multiplicative Systems

This chapter is based on the paper with the same title by Li and Rutkowski [74] published in *Stochastic Processes and their Applications* 122 (2012), 2053–2077.

4.1 Introduction

For an arbitrary random time τ with values in $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \infty$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \bar{\mathbb{R}}_+}$ satisfying the usual conditions, the *Azéma supermartingale* of τ is defined by the equality $G_t^\tau = \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)$ for all $t \in \bar{\mathbb{R}}_+$ (see Azéma [3]). It is clear that the process G^τ is a supermartingale (the càdlàg version) satisfying the inequalities $0 \leq G_t^\tau \leq 1$ for every $t \in \mathbb{R}_+$ with $G_\infty^\tau = 0$ (G_∞^τ should not be confused with $G_{\infty-}^\tau := \lim_{t \to \infty} G_t^\tau$). The Azéma supermartingale of a random time is a central object in the study of the *progressive enlargement* of $\mathbb{F}$ through observations of a random time, since for some classes of random times their probabilistic properties can be characterized in terms the associated Azéma supermartingale (see, e.g., Jeulin [62], Jeulin and Yor [65, 66] and Yor [107]). In particular, the semimartingale decomposition of a $(\mathbb{P}, \mathbb{F})$-semimartingale with respect to the enlarged filtration can sometimes be written using the Azéma supermartingale G^τ or, equivalently, the Azéma submartingale $F^\tau = 1 - G^\tau$. Note that in the following general definition no explicit reference to a random time is made.

Definition 4.1.1. Any (càdlàg) submartingale $F = (F_t)_{t \in \bar{\mathbb{R}}_+}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, satisfying $0 \leq F_t \leq 1$ for every $t \in \mathbb{R}_+$ and $F_\infty = 1$, is called the *Azéma submartingale.*

It is not clear a priori whether a random time associated with an arbitrary Azéma submartingale F exists and how to construct random times associated with F. The goal of this work is to clarify these issues. It is assumed throughout that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \bar{\mathbb{R}}_+}$ satisfying the usual conditions and an Azéma submartingale F are given in advance. By contrast, a random time τ is left unspecified and thus the crucial issue is to provide an explicit construction of τ consistent with the above-mentioned probabilistic data set: the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and the Azéma submartingale F. Formally, by a random time τ *associated with* F, we mean any random time τ, defined on some extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \mathbb{Q})$ of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, such that $\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = F_t$ for all $t \in \bar{\mathbb{R}}_+$ and $\mathbb{Q}$ coincides with $\mathbb{P}$ on $\mathbb{F}$, meaning that the equality $\mathbb{Q}\,|_{\mathcal{F}_t} = \mathbb{P}\,|_{\mathcal{F}_t}$ holds for all $t \in \bar{\mathbb{R}}_+$. The main goal of this work is to show that for any submartingale F satisfying Definition 4.1.1, there exists a random time τ on a *canonical extension* $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\widehat{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$ and F is the Azéma submartingale of τ. We also address the related issue of uniqueness of a $(\mathbb{P}, \mathbb{F})$-conditional distribution associated with an Azéma submartingale under alternative assumptions.

The existence of a random time τ associated with F is well known when F is an increasing process, since in that case it suffices to use the so-called *canonical construction* (see, e.g., Section 8.2.1 in [12]). For a non-trivial case, when F fails to be an increasing process, some preliminary results were established by Gapeev et al. [44] who studied the case of the Brownian filtration $\mathbb{F}$ under the following assumption.

Assumption 4.1.1. The processes (N, Λ) defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfy the following conditions:
(i) the process $(\Lambda_t)_{t\in\mathbb{R}_+}$ is continuous, increasing, such that $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$,
(ii) the process $(N_t)_{t\in\mathbb{R}_+}$ is a positive, continuous $(\mathbb{P}, \mathbb{F})$-martingale such that $N_0 = 1$,
(iii) the submartingale $F := 1 - Ne^{-\Lambda}$ satisfies $F_0 = 0$ and $0 < F_t \leq 1$ for every $t > 0$.

It was shown in Gapeev et al. [44] that, under mild integrability conditions, it is possible to define a probability $\mathbb{Q}$ and a random time τ on an extension of the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\mathbb{Q} = \mathbb{P}$ on $\mathbb{F}$ and F is the Azéma submartingale of τ under $\mathbb{Q}$, that is,

$$\mathbb{Q}\left(\tau \leq \,\middle|\, \mathcal{F}_t\right) = 1 - N_t e^{-\Lambda_t}, \quad \forall\, t \in \bar{\mathbb{R}}_+.$$

The approach developed in [44] hinges on the combination of the canonical construction of a random time associated with am increasing Azéma submartingale $1-e^{-\Lambda}$ with a judiciously chosen equivalent change of a probability measure; hence this method can be referred to as the *change of measure approach*. The present research was directly motivated by the recent papers by Jeanblanc and Song [59, 60] in which the authors proposed another construction of a random time with a given in advance Azéma supermartingale G with known multiplicative decomposition $G = Ne^{-\Lambda}$. Although there is a clear overlap between the present work and [59, 60], the two approaches differ in several respects. First, the method proposed by Jeanblanc and Song [59] focuses on a construction of the probability measure on the canonical space $\Omega \times \mathbb{R}_+$. Second, they start by postulating the hypothesis (*DP*) (see Section 4.2.2) for all $0 \leq u \leq s \leq t$

$$\frac{\mathbb{P}(\tau \leq u \,|\, \mathcal{F}_s)}{\mathbb{P}(\tau \leq s \,|\, \mathcal{F}_s)} \geq \frac{\mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t)}{\mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t)}. \tag{4.1}$$

It is shown in [59] that, under some continuity assumptions on processes N and Λ, condition (4.1) leads to the unique probability measure on the canonical space $\Omega \times \mathbb{R}_+$, which is given by an explicit formula and thus is provides a solution to the problem at hand. Subsequently, Jeanblanc and Song [59] extend their construction to a more general class of Azéma supermartingales, but still maintaining certain technical assumptions (see the assumption $\mathbf{Hy}(N, \Lambda)$ and Theorem 3.1 in [59]).

In contrast to the cases studied in Gapeev et al. [44] and Jeanblanc and Song [59, 60], the Azéma supermartingales associated with random times studied in a recent paper by Nikeghbali and Yor [93] are allowed to hit 1 and stay at that level for an arbitrary length of time. It is thus worth noting that our results cover the class of Azéma supermartingales investigated by Nikeghbali and Yor [93], although the random times considered here and in [93] still have different probabilistic properties. In a recent paper by Kardaras [68], the author examines the problem of universality of the canonical construction of a random time. However, his goal is to study stochastic processes stopped at a random time, whereas in [59, 60], the present work [74] and the follow-up work [75], the emphasis is on explicit constructions of a random time and the study of conditional distributions for non-stopped processes. Hence the paper [68] deals with issues dissimilar from the problems examined in the above mentioned papers. Specifically, the main result in [68] (see Theorem 4.4 therein) shows that, under certain technical assumptions, for the study of probabilistic properties of an $\mathbb{F}$-optional process stopped at a random time τ the canonical construction of τ is essentially sufficient.

This chapter is organized as follows. In Section 4.2, we examine the properties of $(\mathbb{P}, \mathbb{F})$-conditional distributions of random times. We deal there with recently proposed hypotheses (*HP*) (for *proportionality*) and (*DP*) (for *decreasing proportionality*), the (*complete*) *separability* of conditional distributions of a random time, as well as with the non-uniqueness of conditional distributions consistent with a predetermined Azéma submartingale. In Section 4.3, we provide an explicit general construction of a random time τ, given on an extension $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ of the probability space

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, such that the Azéma submartingale of τ coincides with F. A proposed solution is formulated in terms of a *generator* D of F, that is, an increasing (non-adapted) process D defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that F is the $(\mathbb{P}, \mathbb{F})$-optional projection of D. We then solve a similar inverse problem when a $(\mathbb{P}, \mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ of a random time is prescribed, by showing that the general construction can be applied to the increasing process $D_t = F_{t,\infty}$. We have here more a priori information about the probabilistic properties of a random time and we show that the equality $\widehat{\mathbb{P}}(\tau \leq u \,|\, \mathcal{F}_t) = F_{u,t}$ holds for every $u, t \in \bar{\mathbb{R}}_+$. It is also shown, in Section 4.3.3, that our approach can be easily extended to a finite family of correlated random times with predetermined Azéma submartingales. The advantage of the method presented here is that it is more straightforward and general than the constructions developed in Gapeev et al. [44] and Jeanblanc and Song [59].

In Section 4.4, we revisit the concepts of *predictable* and *optional multiplicative systems* associated with a submartingale, introduced by Meyer [85]. Subsequently, in Section 4.5, we present applications of multiplicative systems to constructions of random times. The main goal is to provide an explicit admissible construction of a single random time with a given in advance Azéma submartingale F satisfying Definition 4.1.1. To this end, we first employ the multiplicative system associated with F in order to construct a random field representing the $(\mathbb{P}, \mathbb{F})$-conditional distributions of τ. We show that the construction proposed in Jeanblanc and Song [59] can be obtained as a special case of the multiplicative approach. It is also demonstrated that any honest time can be obtained in that way and the class of honest times can be characterized as the set of all $\mathcal{F}_\infty$-measurable random times that satisfy the hypothesis (*HP*). For further studies of properties of alternative constructions of random times and the corresponding enlargements of filtrations, we refer to the follow-up by Li and Rutkowski [75]. We conclude by providing an example of a non-multiplicative approach to conditional distributions.

4.2 Conditional Distributions of Random Times

In this section, we examine the general properties of the conditional distribution of a random time with respect to some reference filtration. We work throughout on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ that satisfies the usual conditions. We set $\mathcal{F}_{\infty-} = \bigvee_{t\in\mathbb{R}} \mathcal{F}_t$ and we find it convenient to postulate that $\mathcal{F}_\infty = \mathcal{F}_{\infty-}$. We say that $\mathbb{P} = \mathbb{Q}$ on $\mathbb{F}$ whenever the equality $\mathbb{Q}\,|_{\mathcal{F}_t} = \mathbb{P}\,|_{\mathcal{F}_t}$ holds for all $t \in \bar{\mathbb{R}}_+$. Unless explicitly stated otherwise, all stochastic processes are assumed to be càdlàg, but they are not postulated to be $\mathbb{F}$-adapted. For an arbitrary stochastic process X, not necessarily $\mathbb{F}$-adapted, we denote by $\mathbb{F}^X$ the filtration generated by X and satisfying the usual conditions.

4.2.1 Characteristics of Random Times

Let us first introduce the notation for several pertinent characteristics of a random time τ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. We start by defining the *indicator process* $H := \mathbb{1}_{[\![\tau,\infty]\!]}$. Next, we introduce the associated increasing process D, such that $0 \leq D_t \leq 1$ and D_t is $\mathcal{F}_\infty$-measurable, by setting for all $t \in \bar{\mathbb{R}}_+$

$$D_t := \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty) = \mathbb{E}_\mathbb{Q}(H_t \,|\, \mathcal{F}_\infty). \tag{4.2}$$

Finally, we define the $(\mathbb{Q}, \mathbb{F})$-submartingale F associated with τ by setting

$$F_t := \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{E}_\mathbb{Q}(D_t \,|\, \mathcal{F}_t) = \mathbb{E}_\mathbb{P}(D_t \,|\, \mathcal{F}_t)$$

where $\mathbb{P}$ is any probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P} = \mathbb{Q}$ on $\mathbb{F}$. The $(\mathbb{Q}, \mathbb{F})$-supermartingale $G_t = 1 - F_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ is commonly known as the *Azéma supermartingale* of τ. Therefore, we find it natural to refer to the $(\mathbb{Q}, \mathbb{F})$-submartingale F as the *Azéma submartingale* of τ (note that F is also a $(\mathbb{P}, \mathbb{F})$-submartingale).

Definition 4.2.1. The $(\mathbb{Q},\mathbb{F})$-*conditional distribution of* τ is the random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ given by

$$F_{u,t} := \mathbb{Q}(\tau \le u \,|\, \mathcal{F}_t), \quad \forall\, u,t \in \bar{\mathbb{R}}_+.$$

The $(\mathbb{Q},\mathbb{F})$-*conditional survival distribution of* τ is the random field $(G_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ given by

$$G_{u,t} := \mathbb{Q}(\tau > u \,|\, \mathcal{F}_t) = 1 - F_{u,t}, \quad \forall\, u,t \in \bar{\mathbb{R}}_+.$$

Note that the following equalities are valid, for all $t \le u$,

$$F_{u,t} = \mathbb{E}_{\mathbb{Q}}(H_u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(F_u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F_u \,|\, \mathcal{F}_t) \tag{4.3}$$

and thus, in particular, the equality $F_t = F_{t,t}$ holds for all $t \in \bar{\mathbb{R}}_+$. It is also worth noting that $D_t = F_{t,\infty}$ for all $t \in \bar{\mathbb{R}}_+$. It is obvious that the random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ provides more information about the probabilistic properties of a random time then its Azéma submartingale F. We will later argue that the knowledge of a process D conveys more information than the Azéma submartingale F. This is intuitively clear since when a random time τ is not known then F can always be recovered from D, but the converse implication does not hold, in general. It appears that, for a given Azéma submartingale F, one may find several increasing processes D such that D_t is $\mathcal{F}_\infty$-measurable and D generates F through the equality $F_t = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t)$.

4.2.2 Inverse Problems

Generally speaking, by an *inverse problem* we mean a problem of finding random times with some predetermined probabilistic properties with respect to a reference filtration.

Random Time Associated with an Azéma Submartingale

The first inverse problem reads: construct a random time associated with a predetermined Azéma submartingale, where by an *Azéma submartingale* we mean any process F satisfying Definition 4.1.1.

Definition 4.2.2. Let $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}},\mathbb{Q})$ be any extension of the underlying space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ such that $\mathbb{Q} = \mathbb{P}$ on $\mathbb{F}$. A random time *associated with* F is any random variable $\tau : \widetilde{\Omega} \to \bar{\mathbb{R}}_+$ such that $\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = F_t$ for all $t \in \bar{\mathbb{R}}_+$.

If τ can be defined on $(\Omega,\mathcal{F},\mathbb{P})$ as an $\mathcal{F}_\infty$-measurable random variable then one may take $\mathbb{Q} = \mathbb{P}$. However, it is not obvious a priori whether a random time τ can be defined on the original space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, that is, whether the underlying probability space need to be modified.

To analyze the uniqueness of a $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time associated with F, we need first to make precise the concept of $\mathbb{F}$-*equivalence*.

Definition 4.2.3. The random fields $(F^i_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$, $i = 1,2$ are said to be *indistinguishable* if the equality $F^1_{\cdot,\cdot}(\omega) = F^2_{\cdot,\cdot}(\omega)$ holds for almost all ω. Two random times τ^i, $i = 1,2$ given on some extensions $(\widetilde{\Omega}^i,\widetilde{\mathcal{F}}^i,\widetilde{\mathbb{F}}^i,\mathbb{Q}^i)$, $i = 1,2$ of the underlying probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ such that $\mathbb{Q}^1 = \mathbb{Q}^2 = \mathbb{P}$ on $\mathbb{F}$ are said to be $(\mathbb{P},\mathbb{F})$-*equivalent* if the $(\mathbb{P},\mathbb{F})$-conditional distributions of τ^1 and τ^2 are indistinguishable.

Remark 4.2.1. Observe that for any random time τ associated with F and any $t \le u$ we have that

$$\mathbb{Q}(\tau \le u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{Q}(\tau \le u \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}(F_u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F_u \,|\, \mathcal{F}_t).$$

Therefore, in the study of $(\mathbb{P},\mathbb{F})$-equivalence of random times associated with the same Azéma submartingale, it suffices to focus on the conditional probabilities $\mathbb{Q}(\tau \le u \,|\, \mathcal{F}_t)$ for an arbitrary $u \in \mathbb{R}_+$ and all $t \ge u$.

The main motivation for the study of the existence and uniqueness of a random time associated with a predetermined Azéma submartingale comes from the reduced-form approach to credit risk modeling, where it is common to specify explicitly the dynamics of default intensities (i.e., hazard rates) with respect to some reference filtration representing the information flow of market data. In this context, the problem of constructing alternative models of default times for a predetermined family of intensity processes and the study of the properties of a market model arise in a natural way. The first inverse problem is studied here for general Azéma submartingales both for a single random time (see Section 4.3.1) and for a finite family of random times (see Section 4.3.3).

Random Time Consistent with a Conditional Distribution

To formulate the second inverse problem, we need first to introduce the following definition, in which the existence of a random time τ is not postulated a priori. It is clear, however, that the $(\mathbb{P},\mathbb{F})$-conditional distribution of an arbitrary random time τ necessarily satisfies conditions (i)-(iii) of Definition 4.2.4 irrespective of the choice of a reference filtration $\mathbb{F}$.

Definition 4.2.4. A random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ is said to be an $(\mathbb{P},\mathbb{F})$-*conditional distribution* if it satisfies:
(i) for every $u\in\bar{\mathbb{R}}_+$ and $t\in\bar{\mathbb{R}}_+$, we have $0\le F_{u,t}\le 1$, $\mathbb{P}$-a.s.,
(ii) for every $u\in\bar{\mathbb{R}}_+$, the process $(F_{u,t})_{t\in\bar{\mathbb{R}}_+}$ is a $(\mathbb{P},\mathbb{F})$-martingale,
(iii) for every $t\in\bar{\mathbb{R}}_+$, the process $(F_{u,t})_{u\in\bar{\mathbb{R}}_+}$ is right-continuous, increasing and $F_{\infty,t}=1$.
A random field $(G_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is said to be an $(\mathbb{P},\mathbb{F})$-*conditional survival distribution* whenever $F_{u,t}=1-G_{u,t}$ is a $(\mathbb{P},\mathbb{F})$-conditional distribution.

Note that for every $u\in\bar{\mathbb{R}}_+$, conditions (i)-(ii) in Definition 4.2.4 imply that $F_{u,\infty}=\lim_{t\to\infty}F_{u,t}$ and $F_{u,t}=\mathbb{E}_{\mathbb{P}}(F_{u,\infty}\,|\,\mathcal{F}_t)$ for every $t\in\bar{\mathbb{R}}_+$. Since (iii) yields $F_{u,t}\le F_{s,t}$ for all $u\le s$, the (non-adapted) process $(F_{u,\infty})_{u\in\bar{\mathbb{R}}_+}$ is increasing and thus it admits a càdlàg version.

If a $(\mathbb{P},\mathbb{F})$-conditional distribution is given in advance, it is natural to ponder whether one may construct a random time consistent with this distribution. Hence the following inverse problem arises: construct a random time consistent with a predetermined $(\mathbb{P},\mathbb{F})$-conditional distribution, where the consistency is defined as follows.

Definition 4.2.5. A random time τ *consistent with* a $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is any random time τ defined on some extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}},\mathbb{Q})$ of the underlying space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ such that the equality $\mathbb{Q}(\tau\le u\,|\,\mathcal{F}_t)=F_{u,t}$ holds for all $u,t\in\mathbb{R}_+$ and $\mathbb{Q}=\mathbb{P}$ on $\mathbb{F}$.

It is easy to check that for any $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ the process $F_{t,t}$ satisfies Definition 4.1.1 of an Azéma submartingale. In particular, for all $t\le s$

$$\mathbb{E}_{\mathbb{P}}(F_{s,s}\,|\,\mathcal{F}_t)=\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(F_{s,\infty}\,|\,\mathcal{F}_s)\,|\,\mathcal{F}_t\big)\ge\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(F_{t,\infty}\,|\,\mathcal{F}_s)\,|\,\mathcal{F}_t\big)=\mathbb{E}_{\mathbb{P}}(F_{t,\infty}\,|\,\mathcal{F}_t)=F_{t,t}.$$

Any random time consistent with $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is also associated with the Azéma submartingale $F_t:=F_{t,t}$ for all $t\in\bar{\mathbb{R}}_+$. The converse implication does not hold, however, since an Azéma submartingale F does not specify a unique $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$, unless some additional properties are postulated. Hence, to examine the issue of existence and uniqueness of $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ associated with F, we will introduce in Section 4.2.3 some specific classes of $(\mathbb{P},\mathbb{F})$-conditional distributions.

Assume that a random time τ (or, equivalently, its indicator process H) is not observed. Then the Azéma submartingale F can be interpreted as the *signaling process* for the non-observable process H. In a typical situation where the filtration $\mathbb{F}$ is generated by some process X, assumed to be observable, the signaling process F is given as a non-anticipating functional of the sample paths of X. The problem of finding the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ can thus be seen as a non-standard form of the non-linear filtering problem for H when the signaling process equals F. Note also that the process D is based on the observations of X up to infinity, and thus its practical interpretation is less obvious.

Since no explicit model for H is postulated a priori, one can show that a solution $F_{u,t}$ to the abovementioned non-linear filtering problem is not unique, in general (see, for instance, Jeanblanc and Song [59] or Section 4.5.4). We will show, however, that the uniqueness of a solution can be established, provided that some additional assumptions are imposed on the class of considered $(\mathbb{P},\mathbb{F})$-conditional distributions. Furthermore, we also give an explicit construction of τ on the extended probability space (see Section 4.3.2); this provides a fully specified model with a prescribed solution to the linear filtering problem. In other words, one can always find an increasing process H taking values in $\{0,1\}$ such that the equality $F_t = \mathbb{E}_{\mathbb{Q}}(H_t \,|\, \mathcal{F}_t)$ holds for all $t \in \bar{\mathbb{R}}_+$. Although the filtering interpretation may seem artificial at first glance, it proves useful in clarification of previously observed links between the properties of solutions to certain classic non-linear filtering problems and alternative methods of constructing random times with predetermined Azéma submartingales (see, e.g., Gapeev et al. [44] and El Karoui et al. [37]).

4.2.3 Properties of Conditional Distributions

We are now going to analyze some general properties of $(\mathbb{P},\mathbb{F})$-conditional distributions of random times. We first focus on the recently proposed hypotheses (*HP*) and (*DP*). We also recall the classic hypothesis (H), which was studied in numerous papers (see, for instance, Brémaud and Yor [20] or Elliott et al. [35]). Next, we examine the so-called *separability* of a random time representing a $(\mathbb{P},\mathbb{F})$-conditional distribution. Throughout this section, by a $(\mathbb{P},\mathbb{F})$-conditional distribution we mean any random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ satisfying Definition 4.2.4.

Definition 4.2.6. A $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is said to satisfy:
(i) the *hypothesis* (H) whenever for all $0 \le u \le s < t$

$$F_{u,s} = F_{u,t}, \tag{4.4}$$

(ii) the *hypothesis* (*HP*) (or the *proportionality* property) whenever for all $0 \le u < s < t$

$$F_{u,s}F_{s,t} = F_{s,s}F_{u,t}. \tag{4.5}$$

(iii) the *hypothesis* (*DP*) (or the *decreasing proportionality* property) whenever for all $0 \le u < s < t$

$$F_{u,s}F_{t,t} \ge F_{s,s}F_{u,t}. \tag{4.6}$$

Suppose that a random time τ consistent with a $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is given on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ (or on some extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}},\mathbb{Q})$ of this space). Then the hypothesis (*HP*) can also be explained in terms of a peculiar behavior of $\mathbb{F}$-adapted processes with respect to the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ where $\mathbb{H}$ is the natural filtration of the indicator process H. To this end, let us denote by X any $\mathbb{F}$-adapted process given on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, and thus also on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}},\mathbb{Q})$. Then the dynamics of X with respect to the filtration $\mathbb{G}$ have a feature that can be informally stated as follows: the $(\mathbb{P},\mathbb{F})$-conditional distributions of increments of X change when τ occurs, but the dynamics of X after τ are independent of the value of τ.

This property is reminiscent of a similar property of an *honest time* τ, which is manifested by the well-known $\mathbb{G}$-semimartingale decomposition formula for an $\mathbb{F}$-local martingale (see, for instance, Jeulin and Yor [66]). In fact, there is a strong connection between honest times and a larger class of random times satisfying the hypothesis (*HP*). Specifically, any honest time with respect to $\mathbb{F}$ satisfies the hypothesis (*HP*) and any $\mathcal{F}_\infty$-measurable random time satisfying the hypothesis (*HP*) is an honest time (see Proposition 4.5.2).

Example 4.2.1. Assuming that τ models a default time of a firm, the hypothesis (*HP*) has the following interpretation: the dynamics of all $\mathbb{F}$-adapted price processes change at the moment of default, but their behavior after default does not depend on the timing of the default event. It is an open question whether this feature is a desirable property of a credit risk model.

The following result is an immediate consequence of Definition 4.2.6 and thus its proof is omitted.

Lemma 4.2.1. *The following implications are valid:* $(H) \implies (HP) \implies (DP)$.

Remark 4.2.2. The hypothesis (*DP*) was first introduced in the working paper by S. Song in 2009, and it was subsequently termed the *conditional proportionality hypothesis* in Jeanblanc and Song [59]. The hypothesis (*HP*) is termed the *conditional law invariance* in [59]. Note that any $\mathbb{F}$-stopping time manifestly satisfies all three hypotheses of Definition 4.2.6.

Let $F_{u,t} = \mathbb{Q}\left(\tau \le u \,\middle|\, \mathcal{F}_t\right)$ where τ is any random time defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. Then the hypothesis (*HP*) is equivalent to the following equalities, for all $0 \le u < s < t$,

$$\frac{F_{u,s}}{F_{s,s}} = \frac{\mathbb{Q}\left(\tau \le u \,\middle|\, \mathcal{F}_s\right)}{\mathbb{Q}\left(\tau \le s \,\middle|\, \mathcal{F}_s\right)} = \frac{\mathbb{Q}\left(\tau \le u \,\middle|\, \mathcal{F}_t\right)}{\mathbb{Q}\left(\tau \le s \,\middle|\, \mathcal{F}_t\right)} = \frac{F_{u,t}}{F_{s,t}}\,, \tag{4.7}$$

where, by convention, we set $0/0 = 0$. Similarly, the hypothesis (*DP*) can be represented as follows, for all $0 \le u < s < t$,

$$\frac{F_{u,s}}{F_{s,s}} = \frac{\mathbb{Q}\left(\tau \le u \,\middle|\, \mathcal{F}_s\right)}{\mathbb{Q}\left(\tau \le s \,\middle|\, \mathcal{F}_s\right)} \ge \frac{\mathbb{Q}\left(\tau \le u \,\middle|\, \mathcal{F}_t\right)}{\mathbb{Q}\left(\tau \le t \,\middle|\, \mathcal{F}_t\right)} = \frac{F_{u,t}}{F_{t,t}}\,. \tag{4.8}$$

The arguments used in the study of degenerate cases when denominators are equal to zero are the same as in the proof of Lemma 4.2.2 below, so they are not repeated here.

Uniqueness of Conditional Distributions under Hypothesis (*HP*)

We will now show that the uniqueness of conditional distributions holds under Hypothesis (*HP*), provided that some additional technical conditions are met. We will need the following general definition.

Definition 4.2.7. A random field $(C_{u,t})_{u\in\mathbb{R}_+, t\ge u}$ is said to be *multiplicative* if the equality $C_{u,s}C_{s,t} = C_{u,t}$ holds for every $u \le s \le t$.

Lemma 4.2.2. *A* $(\mathbb{P}, \mathbb{F})$*-conditional distribution* $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ *satisfies the hypothesis* (*HP*) *if and only if the random field* $(\widetilde{C}_{u,t})_{u\in\mathbb{R}_+, t\ge u}$ *is multiplicative, where*

$$\widetilde{C}_{u,t} := \frac{F_{u,t}}{F_{t,t}}. \tag{4.9}$$

Proof. Since, by convention, $0/0 = 0$, the random field $(\widetilde{C}_{u,t})_{u\in\mathbb{R}_+, t\ge u}$ given by (4.9) is well defined (since clearly $\{F_{t,t} = 0\} \subset \{F_{u,t} = 0\}$ for $t \ge u$). Let us first show that the hypothesis (*HP*) implies the multiplicative property: $\widetilde{C}_{u,s}\widetilde{C}_{s,t} = \widetilde{C}_{u,t}$. On the event where the denominators are strictly positive, this implication is obvious. It thus remains to consider two degenerate cases, namely, the events $\{F_{s,s} = 0\}$ and $\{F_{t,t} = 0\}$. For the first case, we note that $\widetilde{C}_{u,s} := \frac{F_{u,s}}{F_{s,s}} = 0$ on the event $\{F_{s,s} = 0\}$ since, obviously, $F_{u,s} \le F_{s,s}$ and thus $\{F_{s,s} = 0\} \subset \{F_{u,s} = 0\}$. Hence, if the hypothesis (*HP*) holds, then we get on the event $\{F_{s,s} = 0\}$

$$0 = \widetilde{C}_{u,s}\widetilde{C}_{s,t} = \frac{F_{u,s}}{F_{s,s}}\frac{F_{s,t}}{F_{t,t}} \overset{(HP)}{=} \frac{F_{s,s}F_{u,t}}{F_{s,s}F_{t,t}} = 0 = \frac{F_{u,t}}{F_{t,t}} = \widetilde{C}_{u,t}. \tag{4.10}$$

We also used here the inclusion $A := \{F_{u,s} = 0\} \subset \{F_{u,t} = 0\}$, which holds since A belongs to the $\mathcal{F}_s$ (hence to $\mathcal{F}_t$) so that $0 = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A F_{u,s}) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A F_{u,t})$ where $F_{u,t} \ge 0$ and thus $F_{u,t} = 0$ on A (alternatively, we note that once a positive martingale $F_{u,\cdot}$ hits zero, it will stay at zero). For the second case, we note that the inclusions $\{F_{t,t} = 0\} \subset \{F_{s,t} = 0\} \subset \{F_{u,t} = 0\}$ hold since $F_{u,t} \le F_{s,t} \le F_{t,t}$ for $0 \le u < s < t$. Hence all equalities in (4.10) hold on the event $\{F_{t,t} = 0\}$. To show that the multiplicative property of the random field $\widetilde{C}_{u,t}$ implies the hypothesis (*HP*), we note first that (4.9) and the multiplicative property of $\widetilde{C}_{u,t}$ yield, for all $0 \leq u < s < t$,

$$\frac{F_{u,s}}{F_{s,s}} \frac{F_{s,t}}{F_{t,t}} = \frac{F_{u,t}}{F_{t,t}}.$$

On the event where $F_{t,t}$ and $F_{s,s}$ are strictly positive, we can simply multiply the equality above by $F_{t,t}F_{s,s}$ to show that the hypothesis (*HP*) holds. Let us now consider the degenerate cases. Since $\{F_{s,s} = 0\} \subset \{F_{u,s} = 0\}$, we see that the equality $F_{u,s}F_{s,t} = 0 = F_{s,s}F_{u,t}$ holds trivially on $\{F_{s,s} = 0\}$. Similarly, since $\{F_{t,t} = 0\} \subset \{F_{s,t} = 0\} \subset \{F_{u,t} = 0\}$, we conclude that the equality $F_{u,s}F_{s,t} = 0 = F_{s,s}F_{u,t}$ is satisfied on the event $\{F_{t,t} = 0\}$ as well. □

We are in a position to establish the following uniqueness result.

Proposition 4.2.1. *Let* $(F^i_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$, $i = 1, 2$ *be* $(\mathbb{P}, \mathbb{F})$*-conditional distributions associated with an Azéma submartingale* F*. Assume that:*
(i) the hypothesis (*HP*) *holds for* $F^i_{u,t}$ *for* $i = 1, 2$,
(ii) for any fixed $u \geq 0$*, the* $\mathbb{F}$*-adapted processes* $(\widetilde{C}^i_{u,t})_{t\geq u}$, $i = 1, 2$*, given by*

$$\widetilde{C}^i_{u,t} := \frac{F^i_{u,t}}{F_t}, \tag{4.11}$$

are $(\mathbb{P}, \mathbb{F})$*-predictable.*
Then the $(\mathbb{P}, \mathbb{F})$*-conditional distributions* $F^1_{u,t}$ *and* $F^2_{u,t}$ *are indistinguishable.*

Proof. By assumption, the hypothesis (*HP*) is satisfied by $F^1_{u,t}$ and $F^2_{u,t}$. From Lemma 4.2.2, we deduce that the random fields $\widetilde{C}^1_{u,t}$ and $\widetilde{C}^2_{u,t}$ are multiplicative. From (4.11), it is obvious that $\widetilde{C}^1_{u,t}$ and $\widetilde{C}^2_{u,t}$ are increasing in u. By Lemma 4.2.1, the hypothesis (*HP*) implies the hypothesis (*DP*), which in turn implies that, for a fixed u, the processes $\widetilde{C}^1_{u,t}$ and $\widetilde{C}^2_{u,t}$ are decreasing in t. We also have that, for every $t \in \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{P}}(\widetilde{C}^i_{t,\infty} F_\infty \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\widetilde{C}^i_{t,\infty} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F^i_{t,\infty} \,|\, \mathcal{F}_t) = F^i_{t,t} = F_t.$$

In view of assumption (ii), we conclude that the random fields $\widetilde{C}^1_{u,t}$ and $\widetilde{C}^2_{u,t}$ are two predictable multiplicative systems associated with the $(\mathbb{P}, \mathbb{F})$-submartingale F, in the sense of Definition 4.4.2. It thus follows from Theorem 4.4.2 that the random fields $F^1_{u,t}$ and $F^2_{u,t}$ are indistinguishable. □

Remark 4.2.3. Condition (ii) in Proposition 4.2.1 is necessary, in general. Example 4.5.1 shows that it is possible to construct two different $(\mathbb{P}, \mathbb{F})$-conditional distributions associated with the same Azéma supermartingale and satisfying the hypothesis (*HP*).

Separability of Conditional Distributions

We will now focus on the *separability* property of a $(\mathbb{P}, \mathbb{F})$-conditional distribution. The main goal is to establish the equivalence of some form of separability and the hypothesis (*HP*), under mild technical assumptions. Separability was previously observed for solutions of certain non-linear filtering problems (see, e.g., Section 2.1 in Gapeev et al. [44]). Intuitively, it is related to the fact that the dynamics of an observed process, denoted by U in [44], change when τ occurs, but the $(\mathbb{P}, \mathbb{F})$-conditional distributions of the increments of U after τ do not depend on the value of τ. The following special feature of a $(\mathbb{P}, \mathbb{F})$-conditional distribution was postulated in [44], where the change of measure approach to a construction of a random time was developed.

Definition 4.2.8. We say that a $(\mathbb{P}, \mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is *completely separable* if there exists a positive, $\mathbb{F}$-adapted, increasing process K and a positive $(\mathbb{P}, \mathbb{F})$-martingale L such that $F_{u,t} = K_u L_t$ for every $u, t \in \mathbb{R}_+$ such that $u \leq t$.

It is easily seen that the complete separability of $F_{u,t}$ implies that the hypothesis (HP) holds. Indeed, we have that $F_{u,s}F_{s,t} = (K_u L_s)(K_s L_t) = (K_s L_s)(K_u L_t) = F_{s,s}F_{u,t}$ for all $0 \le u < s < t$. It appears, however, that the property of complete separability is too restrictive, since it does not cover all cases of interest. This motivates the introduction of a weaker notion of *separability*, which will be shown to imply the hypothesis (HP) if a random time is strictly positive.

Definition 4.2.9. We say that a $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is *separable at* $v \ge 0$ if there exist a positive $(\mathbb{P},\mathbb{F})$-martingale $(L^v_t)_{t\in\mathbb{R}_+}$ and a positive, $\mathbb{F}$-adapted, increasing process $(K^v_u)_{u\in[v,\infty)}$ such that the equality $F_{u,t} = K^v_u L^v_t$ holds for every $v \le u \le t$. An $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is called *separable* if it is separable at all $v > 0$.

It is clear from Definitions 4.2.8 and 4.2.9 that a $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is completely separable whenever it is separable at 0.

Lemma 4.2.3. *The following implications hold for any $(\mathbb{P},\mathbb{F})$-conditional distribution:*
(i) if $F_{u,t}$ is separable at $v \ge 0$ then it is also separable at all $s \ge v$,
(ii) if $F_{u,t}$ is separable at $v \ge 0$ then the proportionality property (4.5) holds for all $v \le u < s < t$.

Proof. For part (i), suppose that a $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is separable at v so that, for all $v \le u < t$, we can write $F_{u,t} = K^v_u L^v_t$. By simply setting $L^s_t := L^v_t$ and $K^s_u := K^v_u$, we see that the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is also separable at $s \ge v$. To prove part (ii), suppose that a $(\mathbb{P},\mathbb{F})$-conditional distribution is separable at v. Then the equalities

$$F_{u,s}F_{s,t} = (K^v_u L^v_s)(K^v_s L^v_t) = (K^v_s L^v_s)(K^v_u L^v_t) = F_{s,s}F_{u,t}$$

hold for all $u < s < t$ such that $u \ge v$. □

It appears that the separability of $F_{u,t}$ implies the hypothesis (HP) when $F_0 = 0$, that is, when the random time τ is strictly positive.

Proposition 4.2.2. *If the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is separable and $F_0 = 0$ then the hypothesis (HP) holds.*

Proof. Suppose the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is separable. We wish to show that the proportionality property $F_{u,s}F_{s,t} = F_{s,s}F_{u,t}$ is satisfied for all $0 \le u < s < t$. By part (ii) in Lemma 4.2.3, we only need to consider the case $0 = u \le s \le t$. However, the assumption that $F_0 = 0$ implies that $F_{0,s} = F_{0,t} = 0$ for all $0 < s < t$ and thus $F_{0,s}F_{s,t} = F_{s,s}F_{0,t} = 0$. Hence the proportionality property (4.5) holds for all $0 \le u \le s \le t$, meaning that the hypothesis (HP) is satisfied. □

It is natural to conjecture that in the *non-degenerate case* when $F_{u,t} > 0$ for all $u, t > 0$ the converse implication also holds, that is, if the hypothesis (HP) is valid then $F_{u,t}$ is separable. This is indeed the case, as the following result shows.

Proposition 4.2.3. *Suppose that a non-degenerate $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ satisfies the hypothesis (HP). Then the random field $F_{u,t}$ is separable.*

Proof. Since $F_{u,t}$ satisfies the hypothesis (HP) and is non-degenerate, it can be represented as follows, for all $0 < v \le u < t$,

$$F_{u,t} = \frac{F_{u,u}}{F_{v,u}} F_{v,t} = K^v_u L^v_t,$$

where we set $K^v_u = F_{u,u}(F_{v,u})^{-1}$ and $L^v_t = F_{v,t}$ for every $v \le u \le t$. To conclude that the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is separable at v, it suffices to note that the process K^v is increasing (this follows from (4.6) and Lemma 4.2.1), strictly positive and $\mathbb{F}$-adapted. Also, L^v is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale. □

Although $\{F_{v,u} = 0\} \subset \{F_{v,t} = 0\}$ for $u < t$, the inclusion $\{F_{v,u} = 0\} \subset \{F_{u,t} = 0\}$ need not to hold for $v < u < t$, in general, and thus the non-degeneracy assumption in Proposition 4.2.3 cannot be relaxed. Also, the proof of Proposition 4.2.3 breaks down for $v = 0$ unless we postulate that $F_{0,u} > 0$ for all $u > 0$. However, this assumption is not satisfied if, for instance, τ is a strictly positive random variable and thus it does not seem to be a viable postulate. This supports our claim that the complete separability of a $(\mathbb{P}, \mathbb{F})$-conditional distribution $F_{u,t}$ is less natural assumption to work with than a less restrictive condition of separability of $F_{u,t}$.

4.3 Extended Canonical Constructions

The goal of this section is to present a fairly general method of producing a random time either associated with a predetermined Azéma submartingale or consistent with a given in advance conditional distribution with respect to a reference filtration. It can be viewed as extensions of the classic canonical construction of a random time with an increasing hazard process.

In Section 4.3.1, we will argue that for any Azéma submartingale F the associated (non-unique) random time can be constructed in two steps. In the first step, one has to construct (or select) a generator D of F on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The second step hinges on an ansatz that an increasing process D is the Azéma submartingale with respect to the filtration $\mathbb{F}^D$ of some random time associated with F. One can thus apply the canonical construction of a random time associated with an increasing Azéma submartingale and check that the random time defined in this way is indeed associated with F. Note that the first step can be seen as 'smoothing' of F by enlarging the original filtration $\mathbb{F}$ to the filtration $\mathbb{F}^\infty$ such that $\mathcal{F}_t^\infty = \mathcal{F}_\infty$ for all $t \in \bar{\mathbb{R}}_+$, where the term 'smoothing' refers to the fact that the martingale component of F is formally eliminated through the transition from the submartingale F to the increasing process D such that $F_t = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t)$.

In Section 4.3.2, we present an alternative, but related, approach based on the following steps. For a given Azéma submartingale F, one needs first to construct the associated $(\mathbb{P}, \mathbb{F})$-conditional distribution $F_{u,t}$, as specified in Definition 4.2.4. The next step hinges again on the extended canonical construction based on the increasing process $D_t = F_{t,\infty}$. Note that the $(\mathbb{P}, \mathbb{F})$-conditional distribution $F_{u,t}$ associated with F is typically not unique (see Sections 4.5.2) and indeed it can be specified through an arbitrary mean. For a survey of methods used to obtain the $(\mathbb{P}, \mathbb{F})$-conditional distribution the reader is referred to El Karoui et al. [37], who focus on the so-called *density approach* in which additional regularity of conditional distribution is imposed (see also Filipović et al. [39]).

Finally, in Section 4.3.3, we show that both methods can be easily extended to the case of a finite family $(\tau_1, \dots, \tau_n)$ of random times associated with Azéma submartingales $F^1, \dots, F^n$ and coupled together through an arbitrarily chosen copula function.

It is important to stress that both methods can be implemented using the multiplicative approach to the generator D or, equivalently, to the $(\mathbb{P}, \mathbb{F})$-conditional distribution $F_{u,t}$. We show that by using, in particular, the concept of a predictable multiplicative system associated with a submartingale, we can effectively construct D and/or $F_{u,t}$ for any Azéma submartingale, that is, any process F satisfying Definition 4.1.1.

4.3.1 Random Time with a Predetermined Generator

Let us first recall the concept of a *generator* of a process F satisfying Definition 4.1.1.

Definition 4.3.1. Let $(D_t)_{t \in \bar{\mathbb{R}}_+}$ be an increasing, positive and bounded by 1 process defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that D_t is $\mathcal{F}_\infty$-measurable. We say that D is a *generator* of an Azéma submartingale F whenever F is the $(\mathbb{P}, \mathbb{F})$-optional projection of D, that is, the equality $F_t = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t)$ holds for every $t \in \bar{\mathbb{R}}_+$.

Note that a generator D of F is a non-adapted process, in general. From results of Azéma [4] and Meyer [83, 85], it follows that any positive and bounded submartingale F admits a generator. The well known method of obtaining a generator of F hinges on the notion of a multiplicative system associated with a submartingale; we present this method in Section 4.4.

In the context of the inverse problem formulated in Section 4.2.2, it is natural to conjecture that D_t may be interpreted as the conditional probability $\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty)$, where a yet unspecified probability measure $\mathbb{Q}$ is given on an extended space and coincides with $\mathbb{P}$ on $\mathbb{F}$. The last condition yields immediately the equality $\mathbb{E}_{\mathbb{Q}}(D_t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t) = F_t$, since D_t is $\mathcal{F}_\infty$-measurable.

To construct τ and $\mathbb{Q}$ in a canonical way, we extend the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ by setting $\widehat{\Omega} := \Omega \times [0,1]$, $\widehat{\mathcal{F}} := \mathcal{F} \otimes \mathscr{B}[0,1]$, $\widehat{\mathbb{F}} := \mathbb{F} \otimes \mathscr{B}[0,1]$ and $\widehat{\mathbb{P}} = \mathbb{P} \otimes \lambda$ where λ is the Lebesgue measure on $[0,1]$. We note that the equality $\widehat{\mathbb{P}}\,|_{\mathcal{F}_t} = \mathbb{P}\,|_{\mathcal{F}_t}$ holds for every $t \in \bar{\mathbb{R}}_+$, that is, $\widehat{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$. A uniformly distributed random variable U on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ is given by $U(\widehat{\omega}) = U(\omega, x) = x$. We decided to denote the extension of $\mathbb{P}$ by $\widehat{\mathbb{P}}$, rather than $\mathbb{Q}$, to emphasize the canonical character of construction described in Lemma 4.3.1.

Lemma 4.3.1. *Let F be an arbitrary Azéma submartingale and let D be any generator of F. Then the random time $\tau = \inf\{t \in \mathbb{R}_+ : D_t \geq U\}$, defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$, is associated with F.*

Proof. Since the equality $\{\tau \leq t\} = \{D_t \geq U\}$ is valid, the $\mathcal{F}_\infty$-conditional distribution of τ satisfies

$$\widehat{\mathbb{P}}(\tau \leq t \,|\, \mathcal{F}_\infty) = \widehat{\mathbb{P}}(D_t \geq U \,|\, \mathcal{F}_\infty) = \widehat{\mathbb{P}}(U \leq x)\big|_{x=D_t} = D_t. \tag{4.12}$$

Consequently, for all $t \in \bar{\mathbb{R}}_+$,

$$\widehat{\mathbb{P}}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{E}_{\widehat{\mathbb{P}}}(D_t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t) = F_t.$$

More generally, for all $u \in \bar{\mathbb{R}}_+$,

$$\widehat{\mathbb{P}}(\tau \leq u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(D_u \,|\, \mathcal{F}_t)$$

and thus, in particular, $\widehat{\mathbb{P}}(\tau \leq u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F_u \,|\, \mathcal{F}_t)$ for all $0 \leq t \leq u$. Note that when the underlying probability space is sufficiently rich, so that it supports a uniformly distributed random variable U independent of $\mathcal{F}_\infty$, then τ may be defined on the original space. □

Remark 4.3.1. It is worth pointing out that in any construction of τ associated with an Azéma submartingale F satisfying Definition 4.1.1, the $(\mathbb{P}, \mathbb{F})$-conditional distribution $F_{u,t}$ of the random time τ, for a fixed u and any $t \in [0, u)$, is invariably given by

$$F_{u,t} = \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F_u \,|\, \mathcal{F}_t).$$

Hence the alternative constructions of τ associated with the same F may only differ in the specification of $F_{u,t}$ for $t \in (u, \infty]$.

Remark 4.3.2. It is important to stress that a generator of F is not unique, in general, although we will see that the uniqueness of a generator of F holds if we confine ourselves to generators obtained through predictable multiplicative systems associated with F (see Theorem 4.4.2). In fact, it is justified to focus on the uniqueness of $(\mathbb{P}, \mathbb{F})$-conditional distributions, rather than on the uniqueness of a generator. Once again, Theorem 4.4.2 ensures the former uniqueness if the $(\mathbb{P}, \mathbb{F})$-conditional distribution consistent with F is obtained using predictable multiplicative systems.

In general, the following issue arises: describe the properties of all random times associated with a predetermined process F through the construction proposed in Lemma 4.3.1 for alternative choices of a generator D of F.

4.3.2 Random Time with a Predetermined Conditional Distribution

We now examine a construction of a random time consistent with a predetermined $(\mathbb{P}, \mathbb{F})$-conditional distribution. Note that we do not postulate here that this random time is necessarily consistent with some given in advance Azéma submartingale. The next result is a counterpart of Lemma 4.3.1.

Lemma 4.3.2. *Let $F_{u,t}$ be an arbitrary $(\mathbb{P},\mathbb{F})$-conditional distribution. Then the random time $\tau = \inf\{t \in \mathbb{R}_+ : F_{t,\infty} \geq U\}$, defined on the extension $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, is consistent with $F_{u,t}$.*

Proof. It suffices to apply the construction of Lemma 4.3.1 to the increasing process $F_{t,\infty}$. Hence the equality $\{\tau \leq t\} = \{F_{t,\infty} \geq U\}$ is valid, the $\mathcal{F}_\infty$-conditional distribution of τ satisfies

$$\widehat{\mathbb{P}}\left(\tau \leq t \,\middle|\, \mathcal{F}_\infty\right) = \widehat{\mathbb{P}}\left(F_{t,\infty} \geq U \,\middle|\, \mathcal{F}_\infty\right) = \widehat{\mathbb{P}}(U \leq x)\big|_{x=F_{t,\infty}} = F_{t,\infty}. \tag{4.13}$$

Consequently, using the $(\mathbb{P},\mathbb{F})$-martingale property of $(F_{u,t})_{t\in\bar{\mathbb{R}}_+}$, we obtain

$$\widehat{\mathbb{P}}(\tau \leq u \,|\, \mathcal{F}_t) = \mathbb{E}_{\widehat{\mathbb{P}}}(F_{u,\infty} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(F_{u,\infty} \,|\, \mathcal{F}_t) = F_{u,t}.$$

In particular, the equality $\widehat{\mathbb{P}}\left(\tau \leq t \,\middle|\, \mathcal{F}_t\right) = F_{t,t}$ holds for every $t \in \bar{\mathbb{R}}_+$. □

An admissible construction presented in Subsection 4.2 can be summarized as follows. We start from a supermartingale $(G_t)_{t\in\bar{\mathbb{R}}_+}$ satisfying Assumption 4.1.1 and we first construct from $(G_t)_{t\in\bar{\mathbb{R}}_+}$ the associated $(\mathbb{P},\mathbb{F})$-conditional survival distribution $G_{u,t}$. The random field $G_{u,t}$ is then used to construct a particular random time τ on an extension of the underlying probability space such that the $(\mathbb{P},\mathbb{F})$-conditional survival distribution $G^\tau_{u,t}$ of τ is given by $G_{u,t}$, that is, $\widehat{\mathbb{P}}(\tau > u \,|\, \mathcal{F}_t) = G_{u,t}$.

Remark 4.3.3. A multiplicative system associated with F can also be used to explicitly specify the $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time associated with a predetermined process F (see Lemma 4.5.1). However, other methods are also available for this purpose, so the multiplicative approach to the conditional distribution of a random time is merely one possible option.

Remark 4.3.4. In the case of the *canonical construction* of a random time (see, e.g., Section 8.2.1 in [12]), the $(\mathbb{P},\mathbb{F})$-conditional distribution process $F^\tau_{u,t}$ for $t \in (u,\infty]$ are postulated to satisfy, for every $u < t$,

$$F^\tau_{u,t} = \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_u) = F^\tau_{u,u} = F_u = 1 - G_u.$$

Let us stress that this choice is only possible when the supermartingale G is a decreasing process.

From now on, for a given supermartingale $(G_t)_{t\in\bar{\mathbb{R}}_+}$ satisfying Assumption 4.1.1, we will denote by A the predictable, increasing process which generates G.

4.3.3 Family of Random Times

In this subsection, we assume that we are given a family of positive submartingales F^i for $i = 1, \dots, n$ defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, such that for each i, the submartingale F^i satisfies Definition 4.1.1. For each F^i we denote by $F^i_{u,t}$ any $(\mathbb{P},\mathbb{F})$-conditional distribution consistent with F^i and by D^i a generator of F^i, so that $\mathbb{E}_{\mathbb{P}}(D^i_t \,|\, \mathcal{F}_t) = F^i_t$ for every $i = 1, \dots, n$ and $t \in \bar{\mathbb{R}}_+$.

Since no predetermined dependence structure is imposed a priori on the family $\tau_1, \dots, \tau_n$, there is a lot of flexibility in specifying this dependence. To achieve our goal, we will rely on a common approach based on the concept of the *copula function*. Let us thus denote by $\varphi : [0,1]^n \to [0,1]$ an arbitrary n-dimensional copula function, that is, an n-variate cumulative distribution function on $[0,1]^n$ with uniform marginal distributions, so that, in particular, we have that $\varphi_i(x_i) := \varphi(1, \dots, 1, x_i, 1, \dots, 1) = x_i$ for every $x_i \in [0,1]$ and $i = 1, \dots, n$. In the next result, α^i_u stands either for D^i_u or $F^i_{u,\infty}$.

Proposition 4.3.1. *Assume that we are given a finite family of submartingales F^i, $i = 1, 2, \dots, n$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and satisfying Definition 4.1.1 and an n-dimensional copula function φ. Then there exist random times τ_i, $i = 1, 2, \dots, n$ on an extension $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\widehat{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$ and:*

(i) The joint $(\widehat{\mathbb{P}},\mathbb{F})$*-conditional distribution of* $(\tau_1,\tau_2,\dots,\tau_n)$ *is given by the following expression, for every* $u_1,u_2,\dots,u_n,t\in\mathbb{R}_+$,

$$\widehat{\mathbb{P}}\left(\tau_1\le u_1,\tau_2\le u_2,\dots,\tau_n\le u_n \,\middle|\, \mathcal{F}_t\right)=\mathbb{E}_{\widehat{\mathbb{P}}}\left(\varphi(\alpha^1_{u_1},\alpha^2_{u_2},\dots,\alpha^n_{u_n}) \,\middle|\, \mathcal{F}_t\right).$$

(ii) If $\alpha^i_u=D^i_u$ *then for* $i=1,2,\dots,n$ *and* $0\le t\le u$

$$\widehat{\mathbb{P}}\left(\tau_i\le u \,\middle|\, \mathcal{F}_t\right)=\mathbb{E}_{\mathbb{P}}(F^i_u \,|\, \mathcal{F}_t). \tag{4.14}$$

(iii) If $\alpha^i_u=F^i_{u,\infty}$ *then for* $i=1,2,\dots,n$ *and all* $u,t\in\bar{\mathbb{R}}_+$

$$\widehat{\mathbb{P}}\left(\tau_i\le u \,\middle|\, \mathcal{F}_t\right)=F^i_{u,t}. \tag{4.15}$$

Proof. Since the proof of part (i) for a general n is identical, we will focus on the case of $n=2$. In that case, the copula function φ defines a probability measure μ_φ on the space $([0,1]^2,\mathscr{B}([0,1]^2))$ with uniform marginal probability distributions, specifically, $\mu_\varphi([0,x_1]\times[0,x_2])=\varphi(x_1,x_2)$ for every $(x_1,x_2)\in[0,1]^2$. Let $(U_1(x_1),U_2(x_2))=(x_1,x_2)$ be the identity map on $[0,1]^2$. Then U_i, $i=1,2$ are uniformly distributed random variables on the probability space $([0,1]^2,\mathscr{B}([0,1]^2))$ with the joint distribution μ_φ. We define an extension of the filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ by setting

$$\widehat{\Omega}=\Omega\times[0,1]^2,\quad \widehat{\mathcal{F}}=\mathcal{F}\otimes\mathscr{B}([0,1]^2),\quad \widehat{\mathbb{F}}=\mathbb{F}\otimes\mathscr{B}([0,1]^2),\quad \widehat{\mathbb{P}}=\mathbb{P}\otimes\mu_\varphi.$$

We also formally extend random variables U_i, $i=1,2$, originally given on $([0,1]^2,\mathscr{B}([0,1]^2))$, to random variables on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{\mathbb{P}})$ by setting $U_i(\omega,x_i)=U_i(x_i)$. Therefore, by construction, the random variable (U_1,U_2), which is defined on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{\mathbb{P}})$, is independent of $\mathcal{F}_\infty$ under $\widehat{\mathbb{P}}$. For $i=1,2$, we define the random time τ_i on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{\mathbb{P}})$ by setting $\tau_i=\inf\{u\in\mathbb{R}_+:\alpha^i_u\ge U_i\}$. Then $\{\tau_i\le u_i\}=\{\alpha^i_{u_i}\ge U_i\}$ and thus, for all $u_1,u_2\in\mathbb{R}_+$,

$$\begin{aligned}\widehat{\mathbb{P}}\left(\tau_1\le u_1,\tau_2\le u_2 \,\middle|\, \mathcal{F}_\infty\right)&=\widehat{\mathbb{P}}\left(U_1\le\alpha^1_{u_1},\,U_2\le\alpha^2_{u_2} \,\middle|\, \mathcal{F}_\infty\right)\\&=\widehat{\mathbb{P}}(U_1\le x_1,U_2\le x_2)\big|_{x_1=\alpha^1_{u_1},\,x_2=\alpha^2_{u_2}}\\&=\varphi(\alpha^1_{u_1},\alpha^2_{u_2}).\end{aligned}$$

Consequently, the joint $(\widehat{\mathbb{P}},\mathbb{F})$-conditional distribution of (τ_1,τ_2) is given by the formula

$$\widehat{\mathbb{P}}\left(\tau_1\le u_1,\tau_2\le u_2 \,\middle|\, \mathcal{F}_t\right)=\mathbb{E}_{\widehat{\mathbb{P}}}\big(\varphi(\alpha^1_{u_1},\alpha^2_{u_2}) \,|\, \mathcal{F}_t\big).$$

To establish parts (ii) and (iii), we need also to show that we have the desired marginal conditional distributions. By construction, we have that $\{\tau_2\ge 0\}=\{U_2\le 1\}$ and thus

$$\widehat{\mathbb{P}}\left(\tau_1\le u_1 \,\middle|\, \mathcal{F}_\infty\right)=\widehat{\mathbb{P}}\left(\tau_1\le u_1,\tau_2\ge 0 \,\middle|\, \mathcal{F}_\infty\right)=\widehat{\mathbb{P}}(U_1\le x_1,U_2\le 1)\big|_{x_1=\alpha^1_{u_1}}=\varphi(\alpha^1_{u_1},1)=\alpha^1_{u_1}$$

since $\varphi(x_1,1)=x_1$ for every $x_1\in[0,1]$. By taking the conditional expectation with respect to $\mathcal{F}_t$, we obtain

$$\widehat{\mathbb{P}}\left(\tau_1\le u_1 \,\middle|\, \mathcal{F}_t\right)=\mathbb{E}_{\widehat{\mathbb{P}}}\big(\alpha^1_{u_1} \,|\, \mathcal{F}_t\big).$$

Hence for $\alpha^1_{u_1}=D^1_{u_1}$ we get (4.14), whereas by setting $\alpha^1_{u_1}=F^1_{u_1,\infty}$ we obtain (4.15). □

Example 4.3.1. Let us now examine briefly the $(\mathbb{P},\mathbb{F})$-conditional densities of random times τ_1 and τ_2 constructed in Proposition 4.3.1. We assume here, in particular, that the supermartingales F^1 and F^2 are continuous and never hit one after time zero, that is, they satisfy Assumption 4.5.1. To facilitate computations, we will also make several simplifying assumptions. We start by observing that if a copula function $\varphi(x,y)$ is differentiable then

$$\partial_v\partial_u\widehat{\mathbb{P}}\left(\tau_1\le u,\tau_2\le v \,\middle|\, \mathcal{F}_\infty\right)=\partial_{xy}\varphi(F^1_{u,\infty},F^2_{v,\infty})\partial_u F^1_{u,\infty}\partial_v F^2_{v,\infty}.$$

It now remains to work out the terms $\partial_u F^1_{u,\infty}$ and $\partial_v F^2_{v,\infty}$. Formula (4.34) yields the equality $F^1_{u,\infty} = C_{u,\infty} F^1_\infty$, so that $\partial_u F^1_{u,\infty} = -F^1_\infty \partial_u C^1_{u,\infty}$. Using (4.20), we obtain (note that the processes B^1 and B^2 are continuous)

$$F^1_\infty \partial_u C^1_{u,\infty} = F^1_\infty \partial_u \int_{[0,u]} \frac{C^1_{s,\infty}}{F^1_s}\, dB^1_s = \frac{F^1_\infty C^1_{u,\infty}}{F^1_u}\, \beta^1_u \, du$$

where the second equality holds if we assume, in addition, that the processes B^1 and B^2 are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$, so that $dB^i_t = \beta^i_t \, dt$ for some $\mathbb{F}$-adapted processes β^i, $i = 1, 2$.

Analogous computations can be applied to $\partial_v F^2_{v,\infty}$ and thus the $\mathcal{F}_\infty$-conditional joint distribution of the vector (τ_1, τ_2) satisfies

$$\partial_v \partial_u \widehat{\mathbb{P}}\left(\tau_1 \le u, \tau_2 \le v \,\middle|\, \mathcal{F}_\infty\right) = \partial_{xy}\varphi(F^1_{u,\infty}, F^2_{v,\infty}) \frac{F^1_\infty C^1_{u,\infty}}{F^1_u} \frac{F^1_\infty C^2_{v,\infty}}{F^2_v}\, \beta^1_u \beta^2_v \, du dv.$$

We conclude that the $\mathcal{F}_t$-conditional joint conditional density of τ_1 and τ_2 is given by the following expression

$$\partial_u \partial_v \widehat{\mathbb{P}}\left(\tau_1 \le u, \tau_2 \le v \,\middle|\, \mathcal{F}_t\right) = -\mathbb{E}_{\mathbb{P}} \left(\partial_{xy}\varphi(F^1_{u,\infty}, F^2_{v,\infty}) \frac{F^1_\infty C^1_{u,\infty}}{F^1_u} \frac{F^1_\infty C^2_{v,\infty}}{F^2_v}\, \beta^1_u \beta^2_v \,\middle|\, \mathcal{F}_t \right) du dv$$

provided that

$$\partial_u \partial_v \widehat{\mathbb{P}}\left(\tau_1 \le u, \tau_2 \le v \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}} \left(\partial_u \partial_v \widehat{\mathbb{P}}\left(\tau_1 \le u, \tau_2 \le v \,\middle|\, \mathcal{F}_\infty\right) \,\middle|\, \mathcal{F}_t \right).$$

4.4 Multiplicative Systems

In the *multiplicative approach* to construction of a random time with a given Azéma submartingale, we employ the notion of a *multiplicative system* introduced in Meyer [85] (see also Azéma [4] and Meyer [83]). Hence, for the reader's convenience, we will first summarize the most pertinent definitions and results from [85]. We work here on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}}$ satisfies the usual conditions. As in Section 4.2, we set $\mathcal{F}_{\infty-} = \bigvee_{t \in \mathbb{R}} \mathcal{F}_t$ and we postulate that $\mathcal{F}_\infty = \mathcal{F}_{\infty-}$. We denote by $Y = (Y_t)_{t \in \bar{\mathbb{R}}_+}$ a positive (càdlàg) submartingale and we set $X_t = \mathbb{E}_{\mathbb{P}}(Y_\infty \,|\, \mathcal{F}_t) - Y_t$ for every $t \in \bar{\mathbb{R}}_+$, so that the process $X = (X_t)_{t \in \bar{\mathbb{R}}_+}$ is a positive (càdlàg) supermartingale with $X_\infty = 0$. The process Y is allowed to have a non-negative jump at infinity, specifically,

$$Y_{\infty-} := \lim_{t \to \infty} Y_t \le \lim_{t \to \infty} \mathbb{E}_{\mathbb{P}}\left(Y_\infty \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Y_\infty \,\middle|\, \mathcal{F}_{\infty-}\right) = \mathbb{E}_{\mathbb{P}}\left(Y_\infty \,\middle|\, \mathcal{F}_\infty\right) = Y_\infty, \qquad \mathbb{P}-\text{a.s.}$$

where the penultimate equality follows from Lévy's theorem. By the usual convention, we set $Y_{0-} = 0$ so that Y is continuous at 0 whenever $Y_0 = 0$.

4.4.1 Multiplicative Systems Associated with a Submartingale

The following definition is borrowed from Meyer [85].

Definition 4.4.1. A *multiplicative system* is a positive random field $(C_{u,t})_{u,t \in \bar{\mathbb{R}}_+}$ satisfying the following conditions:
(i) for all $u \le s \le t$ the equality $C_{u,s} C_{s,t} = C_{u,t}$ holds; moreover, $C_{u,t} = 1$ for $u \ge t$,
(ii) for any fixed $u \in \mathbb{R}_+$, the process $(C_{u,t})_{t \in \bar{\mathbb{R}}_+}$ is $\mathbb{F}$-adapted and decreasing,
(iii) for any fixed $t \in \mathbb{R}_+$, the process $(C_{u,t})_{u \in \bar{\mathbb{R}}_+}$ is right-continuous and increasing.
A multiplicative system is called *predictable* (*optional*, resp.) when the process $(C_{u,t})_{t \in \bar{\mathbb{R}}_+}$ is $\mathbb{F}$-predictable ($\mathbb{F}$-optional, resp.).

By convention, we set $C_{0-,t}=0$ for every $t\in\bar{\mathbb{R}}_+$. It should be stressed that the decreasing process $(C_{u,t})_{t\in\bar{\mathbb{R}}_+}$ arising in condition (ii) is not necessarily right- or left-continuous, whereas the increasing process $(C_{u,t})_{u\in\bar{\mathbb{R}}_+}$, defined in condition (iii), is not necessarily $\mathbb{F}$-adapted. We denote by $(D_t)_{t\in\bar{\mathbb{R}}_+}$ the increasing and positive (but non-adapted) process given by $D_t:=C_{t,\infty}$. It is easily seen that D_t is $\mathcal{F}_\infty$-measurable and $0\le D_t\le 1$ for every $t\in\bar{\mathbb{R}}_+$.

Definition 4.4.2. Given a positive submartingale $Y=(Y_t)_{t\in\bar{\mathbb{R}}_+}$, we say that $(C_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is a *multiplicative system associated with* Y if, in addition to conditions (i)-(iii) of Definition 4.4.1, we have, for all $t\in\bar{\mathbb{R}}_+$,

$$\mathbb{E}_{\mathbb{P}}\left(D_tY_\infty\,\middle|\,\mathcal{F}_t\right)=\mathbb{E}_{\mathbb{P}}\left(C_{t,\infty}Y_\infty\,\middle|\,\mathcal{F}_t\right)=Y_t. \tag{4.16}$$

Observe that if $Y_\infty=1$ then (4.16) means that the submartingale Y is the optional projection of D since obviously $\mathbb{E}_{\mathbb{P}}(D_t\,|\,\mathcal{F}_t)=Y_t$ for all $t\in\bar{\mathbb{R}}_+$ (note that here $D_\infty=1$). Hence D is a generator of Y, in the sense of Definition 4.3.1. The following lemma is a rather straightforward consequence of Definitions 4.4.1 and 4.4.2.

Lemma 4.4.1. *Let $(C_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ be a multiplicative system associated with Y. Then, for any fixed $u\in\mathbb{R}_+$, the process $(Q_{u,t}=C_{u,t}Y_t)_{t\in[u,\infty]}$ is a uniformly integrable $(\mathbb{P},\mathbb{F})$-martingale.*

Proof. Let $u\in\mathbb{R}_+$ be fixed and let $t\in[u,\infty]$. Note that the random variable $C_{t,\infty}$ is bounded by 1 and the random variable Y_∞ is integrable (since Y is a submartingale on $\bar{\mathbb{R}}_+$). We multiply both sides of (4.16) by $C_{u,t}$, to get

$$C_{u,t}\,\mathbb{E}_{\mathbb{P}}\left(C_{t,\infty}Y_\infty\,\middle|\,\mathcal{F}_t\right)=C_{u,t}Y_t.$$

Using the properties (i) and (ii) of a multiplicative system, we thus obtain, for every $t\in[u,\infty]$,

$$\mathbb{E}_{\mathbb{P}}\left(Q_{u,\infty}\,\middle|\,\mathcal{F}_t\right)=\mathbb{E}_{\mathbb{P}}\left(C_{u,\infty}Y_\infty\,\middle|\,\mathcal{F}_t\right)=C_{u,t}Y_t=Q_{u,t}.$$

The uniform integrability of the $(\mathbb{P},\mathbb{F})$-martingale $(Q_{u,t})_{t\in[u,\infty]}$ is also easy to check. □

Remark 4.4.1. Since a multiplicative system $C_{u,t}$ associated with a supermartingale Y may not be right- or left-continuous in the second index, we would like to point out that in the rest of the chapter, we always work with the càdlàg version of the $(\mathbb{P},\mathbb{F})$-martingale $Q_{u,t}:=C_{u,t}Y_t$.

4.4.2 Predictable Multiplicative Systems

We will now focus our attention on predictable multiplicative systems. We denote by ${}^pZ$ the $(\mathbb{P},\mathbb{F})$-predictable projection of a process Z.

Lemma 4.4.2. *Let $(\widehat{C}_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ be a predictable multiplicative system associated with Y. Then:*
(i) for any fixed u, the process $(\widehat{C}_{u-,t}Y_t)_{t\in[u,\infty]}$ is a $(\mathbb{P},\mathbb{F})$-martingale,
(ii) the following relationship holds, for every $0\le u<t$,

$$\widehat{C}_{u,t}{}^pY_t=\widehat{C}_{u,t-}Y_{t-}. \tag{4.17}$$

Proof. (i) For any $u\le s\le t$ and $\delta>0$, we have

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{P}}\big(\widehat{C}_{u-\delta,t}Y_t\,|\,\mathcal{F}_s\big)=\lim_{\delta\downarrow 0}\widehat{C}_{u-\delta,s}Y_s=\widehat{C}_{u-,s}Y_s.$$

Recall that the process $\widehat{C}_{u,t}$ is increasing in u. Therefore, by the monotone convergence theorem for conditional expectations, we obtain

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{P}}\big(\widehat{C}_{u-\delta,t}Y_t\,|\,\mathcal{F}_s\big)=\mathbb{E}_{\mathbb{P}}\big(\widehat{C}_{u-,t}Y_t\,|\,\mathcal{F}_s\big).$$

We conclude that the equality $\mathbb{E}_{\mathbb{P}}\big(\widehat{C}_{u-,t}Y_t\,|\,\mathcal{F}_s\big)=\widehat{C}_{u-,s}Y_s$ holds for all $u\le s\le t\le\infty$.

(ii) Let us fix $u \in \mathbb{R}_+$. From Lemma 4.4.1, the process $(\widehat{Q}_{u,t} = \widehat{C}_{u,t} Y_t)_{t \in [u,\infty]}$ is a positive, uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingale. Therefore, using Theorem 4.5 in Nikeghbali [91] in the second equality below, we obtain

$$ {}^p(\widehat{C}_{u,t} Y_t) = {}^p(\widehat{Q}_{u,t}) = \widehat{Q}_{u,t-} = \widehat{C}_{u,t-} Y_{t-}. $$

Moreover, the process $(\widehat{C}_{u,t})_{t \in \bar{\mathbb{R}}_+}$ is $\mathbb{F}$-predictable and thus ${}^p(\widehat{C}_{u,t} Y_t) = \widehat{C}_{u,t} {}^p Y_t$. By combining the above formulae, we obtain (4.17). □

We will also need the following well known property of predictable projection.

Lemma 4.4.3. *Let Y be a positive measurable process such that $Y_t > 0$ for every $t \in \mathbb{R}_+$. Then the $(\mathbb{P}, \mathbb{F})$-predictable projection ${}^pY$ satisfies ${}^pY_t > 0$ for every $t \in \mathbb{R}_+$.*

Proof. The $(\mathbb{P}, \mathbb{F})$-predictable projection ${}^pY$ of Y is defined as the unique (up to indistinguishability) $\mathbb{F}$-predictable process such that the equality

$$ \mathbb{E}_{\mathbb{P}}\big(Y_T \mathbb{1}_{\{T<\infty\}} \,|\, \mathcal{F}_{T-}\big) = {}^pY_T \mathbb{1}_{\{T<\infty\}} $$

holds for every $\mathbb{F}$-predictable stopping time T. By setting $T = t$, we obtain $\mathbb{E}_{\mathbb{P}}(Y_t \,|\, \mathcal{F}_{t-}) = {}^pY_t$ and thus the claim then follows from the properties of the conditional expectation. □

Assume that $Y_\infty = 1$. Then Y is a positive submartingale of class (D) defined on $[0, \infty)$, since the equality $Y_\infty = 1$ implies that $0 \le Y_t \le 1$ for all $t \in \mathbb{R}_+$. Let $Y = -M + A$ be the Doob-Meyer decomposition of Y where M is a uniformly integrable martingale $M_0 = 0$ and A is an $\mathbb{F}$-predictable, increasing process with $A_0 = Y_0$ (see, e.g., Protter [96]). We adopt the convention that $M_\infty = M_{\infty-} := \lim_{t\to\infty} M_t$ so that $M_t = \mathbb{E}_{\mathbb{P}}(M_\infty \,|\, \mathcal{F}_t)$. Consequently, the process A has a jump at infinity of the size equal to the jump of the process Y at infinity, that is, $\Delta A_\infty = \Delta Y_\infty$. Let us finally observe that the supermartingale $X = 1 - Y = 1 + M - A$ is generated by the $\mathbb{F}$-predictable, increasing process A, in the sense that, for all $t \in \bar{\mathbb{R}}_+$,

$$ X_t = \mathbb{E}_{\mathbb{P}}(A_\infty - A_t \,|\, \mathcal{F}_t). \tag{4.18} $$

The following theorem, due to Meyer [85] (see also Azéma [4] for similar results), establishes the existence of an associated predictable multiplicative system for every positive submartingale $Y = (Y_t)_{t \in \bar{\mathbb{R}}_+}$ such that $Y_\infty = 1$.

Theorem 4.4.1. *(i) Any positive submartingale $Y = (Y_t)_{t \in \bar{\mathbb{R}}_+}$ with $Y_\infty = 1$ admits an associated predictable multiplicative system $(\widehat{C}_{u,t})_{u,t \in \bar{\mathbb{R}}_+}$.*
(ii) If $Y_t > 0$ for every $t > 0$ then it suffices to set, for all $0 \le u \le t \le \infty$,

$$ \widehat{C}_{u,t} = \exp\bigg(- \int_{(u,t]} \frac{dA_s^c}{{}^pY_s} \bigg) \prod_{u<s\le t} \bigg(1 - \frac{\Delta A_s}{{}^pY_s} \bigg) \tag{4.19} $$

where ${}^pY$ is the $(\mathbb{P}, \mathbb{F})$-predictable projection of Y, A is the $\mathbb{F}$-predictable, increasing process from the Doob-Meyer decomposition of Y and A^c is the path-by-path continuous component of A.
(iii) For a fixed $u \in \mathbb{R}_+$, the process $(\widehat{C}_{u,t})_{t \in [u,\infty]}$ satisfies $\widehat{C}_{u,u} = 1$ and

$$ d\widehat{C}_{u,t} = -\widehat{C}_{u,t-} ({}^pY_t)^{-1} \, dA_t. \tag{4.20} $$

(iv) For any fixed u, the uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingale $(\widehat{Q}_{u,t} = \widehat{C}_{u,t} Y_t)_{t \in [u,\infty]}$ satisfies

$$ d\widehat{Q}_{u,t} = -\widehat{C}_{u,t} \, dM_t. \tag{4.21} $$

(v) The process $\widehat{D}_t = \widehat{C}_{t,\infty}$ is positive, increasing and bounded by 1. It generates the submartingale Y, in the sense that, for all $t \in \bar{\mathbb{R}}_+$,

$$ Y_t = \mathbb{E}_{\mathbb{P}}(D_t \,|\, \mathcal{F}_t). \tag{4.22} $$

Proof. For the reader's convenience, we present here the sketch of the proof of Theorem 4.4.1 adapted from Meyer [85].

We start by proving part (ii) under an additional assumption that there exists a constant $\epsilon > 0$ such that $Y > \epsilon$. Consequently, the continuous exponential term in (4.19) is manifestly positive and less than one, as the integrand is positive and A is an increasing process. We will now check that all terms of the product are positive and less or equal to one.

To this end, we write $M^A_t = \mathbb{E}_{\mathbb{P}}(A_\infty \,|\, \mathcal{F}_t)$ and $M^Y_t = \mathbb{E}_{\mathbb{P}}(Y_\infty \,|\, \mathcal{F}_t)$. Since M^A and M^Y are uniformly integrable martingales, it is known that their predictable projections satisfy

$$^p(M^A_t) = M^A_{t-}, \quad ^p(M^Y_t) = M^Y_{t-}.$$

We also note that $X = M^A - A$ and $X = M^Y - Y$. Consequently,

$$X_{t-} = M^A_{t-} - A_{t-} = M^Y_{t-} - Y_{t-}$$

and

$$^pX_t = M^A_{t-} - A_t = M^Y_{t-} - {}^pY_t.$$

Simple computations now show that

$$\Delta A_t = {}^pY_t - Y_{t-} \le {}^pY_t. \tag{4.23}$$

We conclude that the jump term in (4.19) is always positive. Moreover, it is strictly positive provided that the positive submartingale Y_- never hits zero.

It is easy to check that the random field $\widehat{C}_{u,t}$ given by (4.19) satisfies conditions (i)–(iii) of Definition 4.4.1 of a multiplicative system. Hence it remains to show that the multiplicative system $\widehat{C}_{u,t}$ is associated with Y. Let us set

$$\mu_t := \widehat{C}_{0,t} = \exp\bigg(- \int_{(0,t]} \frac{dA^c_s}{^pY_s}\bigg) \prod_{0<s\le t} \bigg(1 - \frac{\Delta A_s}{^pY_s}\bigg),$$

so that $\mu_t(\omega)$ defines a predictable measure on $(0,\infty]$ and, in view of (4.20),

$$d\mu_t = -\mu_{t-}({}^pY_t)^{-1}\, dA_t. \tag{4.24}$$

Note also that $\widehat{C}_{t,\infty} = \frac{\mu_\infty}{\mu_t}$. To verify that $\widehat{C}_{u,t}$ is associated with Y, we first observe that

$$\mathbb{E}_{\mathbb{P}}\left(\widehat{C}_{t,\infty} Y_\infty \,\middle|\, \mathcal{F}_t\right) = \frac{1}{\mu_t}\,\mathbb{E}_{\mathbb{P}}\left(\mu_\infty Y_\infty \,\middle|\, \mathcal{F}_t\right) = \frac{1}{\mu_t}\,\mathbb{E}_{\mathbb{P}}\bigg(Y_\infty \mu_t + \int_{(t,\infty]} Y_\infty\, d\mu_s \,\bigg|\, \mathcal{F}_t\bigg).$$

Since $M^Y = Y + M^A - A$, the process $^pY + A_\infty - A$ has the same predictable projection as Y_∞. After substitution, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\widehat{C}_{t,\infty} Y_\infty \,\middle|\, \mathcal{F}_t\right) &= \frac{1}{\mu_t}\,\mathbb{E}_{\mathbb{P}}\bigg(Y_\infty \mu_t + \int_{(t,\infty]} {}^pY_s\, d\mu_s + \int_{(t,\infty]} (A_\infty - A_s)\, d\mu_s \,\bigg|\, \mathcal{F}_t\bigg) \\
&= \frac{1}{\mu_t}\,\mathbb{E}_{\mathbb{P}}\bigg(Y_\infty \mu_t + \int_{(t,\infty]} {}^pY_s\, d\mu_s + \int_{(t,\infty]} \mu_{s-}\, dA_s - (A_\infty - A_t)\mu_t \,\bigg|\, \mathcal{F}_t\bigg)
\end{aligned}$$

where the last equality follows from by an application of the integration by parts formula to $(A_\infty - A_t)\mu_t$. In view of (4.24), the two integrals in the last equation cancel out and thus we arrive at the equality

$$\mathbb{E}_{\mathbb{P}}\left(\widehat{C}_{t,\infty} Y_\infty \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Y_\infty - A_\infty + A_t \,\middle|\, \mathcal{F}_t\right) = Y_t.$$

We will now prove part (i). For the general case, we replace Y by $Y^\epsilon = Y + \epsilon$, so that the generating process A remains unchanged. Let $\widehat{C}^\epsilon_{u,t}$ be given by formula (4.19) with Y replaced by Y^ϵ. Then the following holds

$$\mathbb{E}_{\mathbb{P}}\left(\widehat{C}^\epsilon_{t,\infty}(Y_\infty + \epsilon) \,\middle|\, \mathcal{F}_t\right) = Y_t + \epsilon, \tag{4.25}$$

while $\widehat{C}^{\epsilon}_{t,\infty} \downarrow C_{t,\infty}$, $Y^{\epsilon}_{\infty} \downarrow Y_{\infty}$ and $Y^{\epsilon}_t \downarrow Y_t$ as $\epsilon \downarrow 0$. More generally, $\widehat{C}^{\epsilon}_{u,t} \downarrow C_{u,t}$ as $\epsilon \downarrow 0$ for some random field $C_{u,t}$. It is easy to check that the random field $C_{u,t}$ is a multiplicative system. By the monotone convergence theorem for conditional expectations, the passage to the limit in (4.25) is justified, so that a multiplicative system $C_{u,t}$ is associated with Y.

It remains to establish part (ii) under assumption (b). In the special case when $Y_t > 0$ for every $t > 0$, we start by noting that

$$\lim_{\epsilon\downarrow 0} \widehat{C}^{\epsilon}_{u,t} = \lim_{\epsilon\downarrow 0} \exp\bigg(- \int_{(u,t]} \frac{dA^c_s}{{}^pY_s + \epsilon} \bigg) \prod_{u<s\leq t} \bigg(1 - \frac{\Delta A_s}{{}^pY_s + \epsilon} \bigg).$$

Assume that $0 \leq u \leq t$ and $Y_t > 0$ for every $t > 0$. Then by the a.s. monotone convergence

$$\lim_{\epsilon\downarrow 0} \widehat{C}^{\epsilon}_{u,t} = \exp\bigg(- \int_{(u,t]} \frac{dA^c_s}{{}^pY_s} \bigg) \prod_{u<s\leq t} \bigg(1 - \frac{\Delta A_s}{{}^pY_s} \bigg). \tag{4.26}$$

To obtain equality (4.21), it suffices to apply a version of the Itô formula (see formula (9.3.6) in Jeanblanc et al. [61])

$$d(\widehat{C}_{u,t}Y_t) = \widehat{C}_{u,t}\, dY_t + Y_{t-}\, d\widehat{C}_{u,t} = \widehat{C}_{u,t}\, dY_t - \widehat{C}_{u,t-}Y_{t-}\, ({}^pY_t)^{-1}\, dA_t = \widehat{C}_{u,t}\, d(Y_t - A_t)$$

where we used (4.17) in the last equality. To obtain (4.21), we observe that $d(Y_t - A_t) = -dM_t$. Part (v) is also easy to establish. This completes the proof of the theorem. □

As opposed to the $\mathbb{F}$-predictable process A appearing in (4.18), the process D is not even assumed to be $\mathbb{F}$-adapted. Intuitively, the generator D keeps track of the fluctuations of Y up to infinity and thus it has enough flexibility to be bounded by Y_∞. For the original proof of Theorem 4.4.1, the interested reader is referred to Meyer [85]. Let us only mention that to obtain equality (4.21), it suffices to apply a version of the Itô formula (see formula (9.3.6) in Jeanblanc et al. [61])

$$d(\widehat{C}_{u,t}Y_t) = \widehat{C}_{u,t}\, dY_t + Y_{t-}\, d\widehat{C}_{u,t} = \widehat{C}_{u,t}\, dY_t - \widehat{C}_{u,t-}Y_{t-}\, ({}^pY_t)^{-1}\, dA_t = \widehat{C}_{u,t}\, d(Y_t - A_t)$$

where we used (4.17) in the last equality. To obtain (4.21), we observe that $d(Y_t - A_t) = -dM_t$.

Remark 4.4.2. Just as we postulated that $Y_{0-} = 0$, we will also adopt the convention that $\widehat{Q}_{0-,t} = 0$ for all $t \in \bar{\mathbb{R}}_+$. To see that this convention is consistent with representation (4.19) and the convention that $Y_{0-} = 0$.

It should be stressed that a predictable multiplicative system associated with a positive submartingale Y is not unique, in general. The following result is borrowed from Meyer [85].

Theorem 4.4.2. *Let $\widehat{C}_{u,t}$ and $\widetilde{C}_{u,t}$ be two predictable multiplicative systems associated with a positive submartingale $Y = (Y_t)_{t\in\bar{\mathbb{R}}_+}$. Then the random fields $\widehat{Q}_{u,t} = \widehat{C}_{u,t}Y_t$ and $\widetilde{Q}_{u,t} = \widetilde{C}_{u,t}Y_t$ are indistinguishable. In particular, the generators $\widehat{D}_t = \widehat{C}_{t,\infty}$ and $\widetilde{D}_t = \widetilde{C}_{t,\infty}$ are indistinguishable.*

We will sometimes consider strictly positive submartingales satisfying the following assumption.

Assumption 4.4.1. The positive submartingale $Y = (Y_t)_{t\in\bar{\mathbb{R}}_+}$ with $Y_\infty = 1$ is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and satisfies $Y_t > 0$ and $Y_{t-} > 0$ for every $t > 0$.

The following corollary to Theorems 4.4.1 and 4.4.2 will prove useful in the sequel.

Corollary 4.4.1. *(i) Assume that $Y_t > 0$ for every $t > 0$. Then the random field $\widehat{C}_{u,t}$ given by formula (4.19) is the unique predictable multiplicative system associated with Y.*

(ii) Assume that Y satisfies Assumption 4.4.1. Then the unique predictable multiplicative system associated with Y satisfies the inequality $\widehat{C}_{u,t} > 0$ and equals, for every $0 \le u \le t$,

$$\widehat{C}_{u,t} = \frac{\widehat{C}_{0,t}}{\widehat{C}_{0,u}} = \frac{\mathcal{E}_t \left(- \int_{(0,\cdot]} ({}^pY_s)^{-1} \, dA_s \right)}{\mathcal{E}_u \left(- \int_{(0,\cdot]} ({}^pY_s)^{-1} \, dA_s \right)} \tag{4.27}$$

where $\mathcal{E}_t(U)$ stands for the stochastic exponential of the process U, that is, the unique solution to the stochastic differential equation $d\mathcal{E}_t(U) = \mathcal{E}_{t-}(U) \, ({}^pY_t)^{-1} \, dA_t$ with $\mathcal{E}_0(U) = 1$.

Proof. From Corollary 4.4.1, we know that $\widehat{C}_{u,t} > 0$ for all $u, t \in \bar{\mathbb{R}}_+$ and thus (4.27) is an immediate consequence of (4.19). □

As already mentioned, the uniqueness of a predictable multiplicative system associated with a positive submartingale Y fails to hold, in general. However, from Theorem 3 in Meyer [85], restated here as Theorem 4.4.3, it follows that the uniqueness holds if we restrict our attention to the class of predictable multiplicative systems associated with Y satisfying the following additional property.

Property (A). For any $t \in \mathbb{R}_+$, if the equality $Y_{t-} = 0$ holds then $\widehat{C}_{u,t} = 0$ for every $u < t$.

The property (A) implies that if $u < t$ and if there exists $s \in (u, t]$ such that $Y_{s-} = 0$ then $\widehat{C}_{u,t} = 0$.

Theorem 4.4.3. *There exists a unique predictable multiplicative system associated with $Y = (Y_t)_{t \in \bar{\mathbb{R}}_+}$ such that the property (A) holds.*

We observe that formula (4.19) yields

$$\widehat{C}_{u,t} = \exp \left(- \int_{(u,t]} \frac{dA^c_s}{{}^pY_s} \right) \prod_{u < s \le t} \left(\frac{Y_{s-}}{{}^pY_s} \mathbb{1}_{\{Y_{s-} \neq {}^pY_s\}} \right), \tag{4.28}$$

so that the random field given by (4.19) satisfies the property (A). It is thus natural to conjecture that for any positive submartingale Y the unique predictable multiplicative system associated with Y and enjoying the property (A) is given by the formula

$$\widehat{C}_{u,t} = \begin{cases} 0, & u \in [0, L_t), \\ \exp \left(- \int_{(u,t]} \frac{dA^c_s}{{}^pY_s} \right) \prod_{u < s \le t} \left(\frac{Y_{s-}}{{}^pY_s} \mathbb{1}_{\{Y_{s-} \neq {}^pY_s\}} \right), & u \in [L_t, t]. \end{cases} \tag{4.29}$$

Optional Multiplicative Systems

Inspired by Meyer [85] and Kardaras in [68, 69], we will now demonstrate the existence of an optional multiplicative system $\bar{C}_{u,t}$ associated with the Azéma submartingale of a random time. Suppose that we are given a random time $\widehat{\tau} : \Omega \to \bar{\mathbb{R}}_+$ defined on the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\bar{A}$ the dual $(\mathbb{P}, \mathbb{F})$-optional projection of a (non-$\mathbb{F}$-adapted) process $H := \mathbb{1}_{[\![\widehat{\tau}, \infty]\!]}$. The $(\mathbb{P}, \mathbb{F})$-optional projection of H is denoted by F, so that $F_t = \mathbb{P}(\widehat{\tau} \le t \,|\, \mathcal{F}_t)$. Finally, V stands for the $(\mathbb{P}, \mathbb{F})$-optional projection of $\mathbb{1}_{[\![0, \widehat{\tau}]\!]}$. At infinity, we have that $F_\infty = 1$, whereas for the dual $(\mathbb{P}, \mathbb{F})$-optional projection we set $\bar{A}_\infty := \bar{A}_{\infty-} + \Delta F_\infty$.

Let us recall the following well known result.

Proposition 4.4.1. *The process G is generated by $\bar{A}$, that is, for all $t \ge 0$*

$$G_t = \mathbb{E}_{\mathbb{P}}(\bar{A}_\infty - \bar{A}_t \,|\, \mathcal{F}_t) = \bar{M}_t - \bar{A}_t$$

where the positive $(\mathbb{P}, \mathbb{F})$-martingale $\bar{M}$ is defined by $\bar{M}_t := \mathbb{E}_{\mathbb{P}}(\bar{A}_\infty \,|\, \mathcal{F}_t)$.

Our aim is to show the existence of an optional multiplicative system associated with the submartingale F. We first consider the case of a strictly positive submartingale F.

Lemma 4.4.4. *Assume that the submartingale $F_t = \mathbb{P}\left(\tau \le t \,\middle|\, \mathcal{F}_t\right)$ is bounded below by a strictly positive constant. Let the random field $\bar{C}_{u,t}$ be defined by $\bar{C}_{u,t} = 1$ for all $u \ge t$ and*

$$d\bar{C}_{u,t} = -\bar{C}_{u,t-}(F_t)^{-1}\, d\bar{A}_t, \quad \forall\, t \ge u. \tag{4.30}$$

Then the random field $\bar{C}_{u,t}$ is an optional multiplicative system and, for any fixed $u \ge 0$, the process $(\bar{Q}_{u,t} := \bar{C}_{u,t}F_t)_{t\in[u,\infty]}$ is a positive $(\mathbb{P},\mathbb{F})$-martingale bounded by 1 and it satisfies

$$d\bar{Q}_{u,t} = -\bar{C}_{u,t-}\, d\bar{M}_t. \tag{4.31}$$

Proof. We will first show that, for any fixed u, the process $(\bar{C}_{u,t})_{t\in[0,\infty]}$ is positive and bounded by 1. To this end, it suffices to observe that it is a decreasing process with $\bar{C}_{u,t} = 1$ for $t \le u$ and with the jump at t satisfying $\bar{C}_{u,t} = \bar{C}_{u,t-}(1 - F_t^{-1}\Delta\bar{A}_t)$ for $t \ge u$. Since $G_t = V_t - \Delta\bar{A}_t$ (see Jeulin [62], page 576), we obtain

$$0 \le (1 - V_t)F_t^{-1} = 1 - F_t^{-1}\Delta\bar{A}_t \le 1,$$

and thus we conclude that the process $(\bar{C}_{u,t})_{t\in[0,\infty]}$ is positive and bounded by 1. Therefore, the processes $(\bar{Q}_{u,t} = \bar{C}_{u,t}F_t)_{t\in[u,\infty]}$ are positive and bounded by 1. Next, we show that the process $(\bar{Q}_{u,t})_{t\in[u,\infty]}$ is an $\mathbb{F}$-martingale. To this end, we observe that formula (9.3.5) in Jeanblanc et al. [61] yields

$$d\bar{Q}_{u,t} = \bar{C}_{u,t-}\, dF_t + F_t\, d\bar{C}_{u,t} = \bar{C}_{u,t-}\, d(F_t - \bar{A}_t) = -\bar{C}_{u,t-}\, d\bar{M}_t$$

since $d(F_t - \bar{A}_t) = -d\bar{M}_t$. Since the process $(\bar{Q}_{u,t})_{t\in[u,\infty]}$ is bounded by 1, we conclude by recalling that a bounded local martingale is a martingale. □

Remark 4.4.3. For every $u \ge 0$, the process $(\bar{Q}_{u,t})_{t\in[u,\infty]}$ is a positive supermartingale and the equality $\lim_{t\to\infty} \bar{Q}_{u,t} = \bar{C}_{u,\infty-}F_{\infty-}$ holds. One can check that, under our convention, for every $u \ge 0$ the $(\mathbb{P},\mathbb{F})$-martingale $(\bar{Q}_{u,t})_{t\in[u,\infty]}$ is continuous at infinity, since $\Delta\bar{M}_\infty = 0$ and thus the equality $\Delta\bar{Q}_{u,\infty} = -\bar{C}_{u,\infty-}\Delta\bar{M}_\infty = 0$ is satisfied for every $u \ge 0$.

We are ready to establish the existence of an optional multiplicative system associated with the submartingale F.

Corollary 4.4.2. *There exists an optional multiplicative system associated with $F_t = \mathbb{P}\left(\tau \le t \,\middle|\, \mathcal{F}_t\right)$.*

Proof. We set $\bar{C}_{u,t} := \lim_{\epsilon\downarrow 0} \bar{C}^{\epsilon}_{u,t}$ where $\bar{C}^{\epsilon}_{u,t}$ is defined in Lemma 4.4.4 for $F^\epsilon = F + \epsilon$. It is obvious that the process $\bar{C}^{\epsilon}_{u,t}$ is decreasing in t and increasing in u. Moreover,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(F_\infty\bar{C}_{u,\infty} \,\middle|\, \mathcal{F}_u\right) &= \mathbb{E}_{\mathbb{P}}\left(\lim_{\epsilon\downarrow 0}\, (F_\infty + \epsilon)\bar{C}^{\epsilon}_{u,\infty} \,\middle|\, \mathcal{F}_u\right) = \lim_{\epsilon\downarrow 0} \mathbb{E}_{\mathbb{P}}\left(F^{\epsilon}_\infty\bar{C}^{\epsilon}_{u,\infty} \,\middle|\, \mathcal{F}_u\right)\\ &= \lim_{\epsilon\downarrow 0}\, (F_u + \epsilon) = F_u\end{aligned}$$

where the second equality holds by the monotone convergence theorem and the third equality is a consequence of Lemma 4.4.4. □

Remark 4.4.4. For a given in advance Azéma submartingale F, the random time $\widehat{\tau}$ may be constructed using the predictable multiplicative system $\widehat{C}_{u,t}$ associated with F and following the method presented in Section 4.5. Subsequently, one may define an optional multiplicative system associated with F using Proposition 4.4.4 and Corollary 4.4.2, and construct another random time associated with F through the method of Section 4.5. Then both random times will have a common Azéma submartingale, but their $(\mathbb{P},\mathbb{F})$-conditional distribution will differ, in general.

4.5 Multiplicative Approach to Random Times

Our next goal is to apply the concepts and results presented in preceding sections to establish the existence of an explicit solution to the problem of a construction of a random time with a predetermined Azéma submartingale. We first prove that any multiplicative system associated with F can be used to define a $(\mathbb{P},\mathbb{F})$-conditional distribution associated with F and the resulting $(\mathbb{P},\mathbb{F})$-conditional distribution always satisfies the hypothesis (HP). In Section 4.5.2, we show that the uniqueness of a $(\mathbb{P},\mathbb{F})$-conditional distribution obtained in this way does not hold. If, however, we chose any predictable multiplicative system associated with F, then the generator D of F, as well as the $(\mathbb{P},\mathbb{F})$-conditional distribution $\widehat{F}_{u,t}$ associated with F are unique and, under mild technical assumption, the random field $\widehat{F}_{u,t}$ is completely separable. The chapter concludes with a brief analysis of *honest times* (Section 4.5.3) and an example of the non-multiplicative approach (Section 4.5.4).

4.5.1 Multiplicative Construction of a Random Time

Let F be an arbitrary Azéma submartingale. We will now show that a multiplicative system associated with F can be used to construct the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ such that $F_{t,t}=F_t$ for all $t\in\bar{\mathbb{R}}_+$.

Lemma 4.5.1. *Let F be an arbitrary Azéma submartingale and let $(C_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ be any multiplicative system associated with F. We define the random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ by setting*

$$F_{u,t}=\begin{cases}\mathbb{E}_{\mathbb{P}}\left(F_u \,\middle|\, \mathcal{F}_t\right), & t\in[0,u),\\ C_{u,t}F_t, & t\in[u,\infty].\end{cases} \tag{4.32}$$

Then $F_{u,t}$ satisfies Definition 4.2.4 of a $(\mathbb{P},\mathbb{F})$-conditional distribution and $F_{t,t}=F_t$. Moreover, the hypothesis (HP) holds.

Proof. The equality $F_{t,t}=F_t$ is obvious since $C_{t,t}=1$. Therefore, it suffices to show that $F_{u,t}$ satisfies conditions (i)–(iii) of Definition 4.2.4.

Condition (i). Let us first check that $0\le F_{u,t}\le 1$. This is obvious for $t<u$ since $0\le F\le 1$. For $t\ge u$, it is enough to use the properties of the multiplicative system. Recall that $C_{u,t}$ is increasing in u and $C_{u,t}=1$ if $u\ge t$ and the process F satisfies $0\le F\le 1$.
Condition (ii). Let us fix u. It suffices to check this condition for $u\le t$, since it is obvious for $t\le u$ and there is no jump at u. By Lemma 4.4.1, the process $C_{u,t}F_t$ is a martingale for $t\ge u$.
Condition (iii). Let us fix t. We first observe that the process $(F_{u,t})_{t\le u}$ is increasing in u. Indeed, F is a submartingale and thus, for all $t\le u\le s$

$$F_{u,t}=\mathbb{E}_{\mathbb{P}}\left(F_u \,\middle|\, \mathcal{F}_t\right)\le\ \mathbb{E}_{\mathbb{P}}\left(F_s \,\middle|\, \mathcal{F}_t\right)=F_{s,t}.$$

Furthermore, condition (iii) in Definition 4.4.1 of a multiplicative system implies that the process $(F_{u,t})_{t\ge u}$ is increasing in u. We also have that $F_{\infty,t}=1$ since $C_{u,t}=1$ for $u\ge t$ and $F_\infty=1$. Since $C_{u,t}$ is a multiplicative random field, we obtain, for all $0\le u<s<t$,

$$\frac{F_{u,t}}{F_{t,t}}=C_{u,t}=C_{u,s}C_{s,t}=\frac{F_{u,s}}{F_{s,s}}\frac{F_{s,t}}{F_{t,t}}. \tag{4.33}$$

We thus see from (4.5) that the hypothesis (HP) is indeed satisfied. □

We are in a position to establish the result, which furnishes an explicit solution to the problem of construction of τ associated with a predetermined Azéma submartingale F. To prove Theorem 4.5.1, it suffices to combine Lemma 4.3.2 and Lemma 4.5.1.

Theorem 4.5.1. *Let F be any Azéma submartingale and let $C_{u,t}$ be any multiplicative system associated with F. Then there exists a random time τ on the extension $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that the $(\mathbb{P}, \mathbb{F})$-conditional distribution of τ under $\widehat{\mathbb{P}}$ equals*

$$F_{u,t} = \widehat{\mathbb{P}}(\tau \le u \,|\, \mathcal{F}_t) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(F_u \,\middle|\, \mathcal{F}_t\right), & t \in [0, u), \\ C_{u,t} F_t, & t \in [u, \infty]. \end{cases} \tag{4.34}$$

In particular, the equality $\widehat{\mathbb{P}}\left(\tau \le t \,\middle|\, \mathcal{F}_t\right) = F_t$ holds for every $t \in \bar{\mathbb{R}}_+$ and $\widehat{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$.

Recall that Theorem 4.4.1 ensures the existence of a predictable multiplicative system associated with F. It is also evident from Theorem 4.4.2 that, although a predictable multiplicative system associated with the positive submartingale F may not be unique, the $(\mathbb{P}, \mathbb{F})$-conditional distribution $\widehat{F}_{u,t}$ constructed from any predictable multiplicative systems $\widehat{C}_{u,t}$ associated with F is unique and satisfies, for all $0 \le u \le t$, the equality $\widehat{F}_{u,t} = \widehat{C}_{u,t} F_t =: \widehat{Q}_{u,t}$ where, by Lemma 4.4.1, for a fixed $u \in \mathbb{R}_+$, the process $(\widehat{Q}_{u,t})_{t \ge u}$ is a uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingale. Let us also observe that in Theorem 4.5.1 we only need to deal with the random field $(C_{u,t})_{u \in \mathbb{R}_+,\, t \ge u}$ since for $u = \infty$ we always have that $\widehat{F}_{\infty,t} = \widehat{C}_{\infty,t} = 1$.

In the case of the predictable multiplicative approach to an admissible construction of a random time τ, the $(\mathbb{P}, \mathbb{F})$-conditional distribution $\widehat{F}_{u,t}$

Corollary 4.5.1. *The random time τ constructed in Theorem 4.5.1 is with probability 1:*
(i) a finite random time if and only if F is continuous at infinity, that is, the equality $F_\infty = F_{\infty-}$ holds.
(ii) a strictly positive random time if and only if F is continuous at time zero, that is, $F_0 = F_{0-} = 0$.

Proof. To prove (i), we observe that (4.13) yields

$$\widehat{\mathbb{P}}\left(\tau = \infty \,\middle|\, \mathcal{F}_\infty\right) = \lim_{u \to \infty} \widehat{\mathbb{P}}\left(\tau > u \,\middle|\, \mathcal{F}_\infty\right) = \lim_{u \to \infty} (1 - F_{u,\infty}) = 1 - F_{\infty-,\infty} = 1 - C_{\infty-,\infty} F_\infty = 1 - C_{\infty-,\infty}.$$

Since $F_\infty = 1$ and $C_{\infty-,\infty} = F_{\infty-} \le 1$, we conclude that the equality $\widehat{\mathbb{P}}(\tau = \infty) = 0$ holds whenever $F_\infty = F_{\infty-}$. For part (ii), we note that $\widehat{\mathbb{P}}\left(\tau = 0 \,\middle|\, \mathcal{F}_0\right) = \widehat{\mathbb{P}}\left(\tau \le 0 \,\middle|\, \mathcal{F}_0\right) = F_0 > 0$. This implies that $\widehat{\mathbb{P}}(\tau = 0) = \mathbb{E}_{\mathbb{P}}(F_0)$ and thus $\widehat{\mathbb{P}}(\tau = 0) > 0$ whenever $\mathbb{P}(F_0 > 0) = \mathbb{P}(F_0 \ne F_{0-}) > 0$. □

The following assumption corresponds to Assumption 4.4.1.

Assumption 4.5.1. The submartingale $F = (F_t)_{t \in \bar{\mathbb{R}}_+}$ with $F_\infty = 1$, given on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, is such that $F_t > 0$ and $F_{t-} > 0$ for every $t > 0$.

Recall that the equality $F = -M + A$ is the Doob-Meyer decomposition of the bounded $(\mathbb{P}, \mathbb{F})$-submartingale F.

Corollary 4.5.2. *Under Assumption 4.5.1, the unique predictable multiplicative system associated with F is given by*

$$\widehat{C}_{u,t} = \frac{\widehat{C}_{0,t}}{\widehat{C}_{0,u}} = \frac{\mathcal{E}_t\left(-\int_{(0,\cdot]} ({}^pF_s)^{-1}\, dA_s\right)}{\mathcal{E}_u\left(-\int_{(0,\cdot]} ({}^pF_s)^{-1}\, dA_s\right)} \tag{4.35}$$

and the random field $\widehat{F}_{u,t}$ is strictly positive, except perhaps for $\widehat{F}_{0,0} = F_0$.

Proof. Formula (4.35) is an immediate consequence of (4.27). Moreover, from part (ii) in Corollary 4.4.1, we see that $\widehat{C}_{u,t} > 0$ for all $u \le t$ and thus $\widehat{F}_{u,t} = \widehat{C}_{u,t} F_t > 0$ for all $u \le t,\ t > 0$. Also $\widehat{F}_{u,t} = \mathbb{E}_{\mathbb{P}}\left(F_u \,\middle|\, \mathcal{F}_t\right) > 0$ for all $u > t$. Hence the random field $\widehat{F}_{u,t}$ is strictly positive, except perhaps for $\widehat{F}_{0,0} = F_0$. □

The following result shows that $\widehat{F}_{u,t}$ is completely separable if Assumption 4.5.1 is satisfied.

Proposition 4.5.1. *Under Assumption 4.5.1, let the* $(\mathbb{P},\mathbb{F})$*-conditional distribution* $\widehat{F}_{u,t} = \widehat{\mathbb{P}}(\tau \leq u \,|\, \mathcal{F}_t)$ *be given by (4.34) with the predictable multiplicative system* $\widehat{C}_{u,t}$ *given by (4.35). Then the random field* $\widehat{F}_{u,t} = \widehat{C}_{u,t}F_t$ *is completely separable, namely, for all* $0 \leq u \leq t$,

$$\widehat{F}_{u,t} = F_t \,\frac{\widehat{C}_{0,t}}{\widehat{C}_{0,u}} = \frac{\widehat{Q}_{0,t}}{\widehat{C}_{0,u}} = K_u L_t$$

where the strictly positive, $\mathbb{F}$*-adapted, increasing process* K *is given by*

$$K_u = (\widehat{C}_{0,u})^{-1} = \left[\mathcal{E}_u\bigg(- \int_{(0,\cdot]} ({}^p F_s)^{-1}\, dA_s\bigg)\right]^{-1}$$

and the strictly positive, uniformly integrable $(\mathbb{P},\mathbb{F})$*-martingale* L *equals*

$$L_t = \widehat{Q}_{0,t} = F_t \,\mathcal{E}_t\bigg(- \int_{(0,\cdot]} ({}^p F_s)^{-1}\, dA_s\bigg). \tag{4.36}$$

Remark 4.5.1. It is preferable to work under Assumption 4.5.1, since it allows us to write down explicitly the form of the predictable multiplicative system associated with F, which is desirable in applications. Moreover, it also allows one to see that, under Assumption 4.5.1, the process $(\widehat{F}_{u,t})_{t\geq u} := \widehat{C}_{u,t}F_t$ coincides with the basic iM_Z introduced in Jeanblanc and Song [60]. In view of Lemma 4.2.1, the pair $(\tau,\widehat{\mathbb{P}})$ constructed in Theorem 4.5.1 satisfies the hypothesis (DP) as well, whereas it is clear that the hypothesis (H) need not to be fulfilled by the pair $(\tau,\widehat{\mathbb{P}})$, in general. Furthermore, since a multiplicative system $C_{u,t}$ is decreasing in t, we obtain directly, for all $0 \leq u < s < t$,

$$\frac{\widehat{\mathbb{P}}\left(\tau \leq u \,\middle|\, \mathcal{F}_s\right)}{\widehat{\mathbb{P}}\left(\tau \leq s \,\middle|\, \mathcal{F}_s\right)} = \frac{F_{u,s}}{F_{s,s}} = C_{u,s} \geq C_{u,t} = \frac{F_{u,t}}{F_{t,t}} = \frac{\widehat{\mathbb{P}}\left(\tau \leq u \,\middle|\, \mathcal{F}_t\right)}{\widehat{\mathbb{P}}\left(\tau \leq t \,\middle|\, \mathcal{F}_t\right)}.$$

The last observation can be seen as an independent motivation for the hypothesis (DP). In fact, in the paper by Jeanblanc and Song [59], the hypothesis (DP) is postulated a priori and is subsequently used, under some technical assumptions on the process $G = Ne^{-\Lambda}$, to obtain a solution to the problem of existence of a random time with a predetermined Azéma supermartingale G.

4.5.2 Non-Uniqueness of Conditional Distributions

We now show, by means of a counterexample, that condition (ii) in Proposition 4.2.1 is necessary. We claim that it is possible to construct two random times associated the same Azéma submartingale and satisfying hypothesis (HP), but with different $(\mathbb{P},\mathbb{F})$-conditional distributions. To justify this statement, let us assume that we are given a strictly positive Azéma submartingale F with $F_0 = 0$. By applying Theorem 4.5.1 to a predictable multiplicative system $\widehat{C}_{u,t}$, we obtain the existence of a random time τ satisfying the hypothesis (HP). Given the random time τ, we can apply Corollary 4.4.2 to show the existence of an optional (non-predictable) multiplicative system $\bar{C}_{u,t}$ associated with F. Therefore, by an application of Theorem 4.5.1 to $\bar{C}_{u,t}$, we obtain the existence of another random time, $\bar{\tau}$ say, such that:
(i) the process F is the Azéma submartingale of τ and $\bar{\tau}$,
(ii) the random times τ and $\bar{\tau}$ satisfy the hypothesis (HP),
(iii) the $(\mathbb{P},\mathbb{F})$-conditional distributions of τ and $\bar{\tau}$ are different, in general.

Property (iii) holds since the $(\mathbb{P},\mathbb{F})$-conditional distributions of τ and $\bar{\tau}$ are given by $\widehat{F}_{u,t} = \widehat{C}_{u,t}F_t = \widehat{Q}_{u,t}$ and $\bar{F}_{u,t} = \bar{C}_{u,t}F_t = \bar{Q}_{u,t}$ for all $u \leq t$, where in turn (see (4.21) and (4.31))

$$d\widehat{Q}_{u,t} = -\widehat{C}_{u,t}\, dM_t, \qquad d\bar{Q}_{u,t} = -\bar{C}_{u,t-}\, d\bar{M}_t. \tag{4.37}$$

Equivalently, in view of (4.20) and (4.30)

$$\widehat{C}_{u,t} = 1 - \int_{(u,t]} \frac{\widehat{C}_{u,s-}}{{}^pF_s}\, dA^\tau_s, \qquad \bar{C}_{u,t} = 1 - \int_{(u,t]} \frac{\bar{C}_{u,s-}}{F_s}\, d\bar{A}^\tau_s$$

where $\bar{A}^\tau$ and A^τ denote the dual $\mathbb{F}$-optional and $\mathbb{F}$-predictable projections of $\mathbb{1}_{[\![\tau,\infty[\![}$. Sadly, it is not easy to give an explicit non-trivial example of a random time for which the dual optional projection is known and thus is it more convenient to focus instead on dynamics (4.37).

Example 4.5.1. To give an explicit (albeit somewhat artificial) counterexample, suppose that T_1 is the first jump time of a Poisson process with intensity λ. Let the filtration $\mathbb{F}$ be generated by the process $H^1_t = \mathbb{1}_{\{T_1 \le t\}}$. Then, for any $0 < \epsilon < 1$, the process $F_t = 1 - (1-\epsilon)(1 - H^1_t)$ is strictly positive, increasing and satisfies Definition 4.1.1. On the one hand, it is easy to check that the random field $\bar{C}_{u,t} := F_u F_t^{-1}$ for $u \le t$ is an optional multiplicative system associated with F. By an application of Theorem 4.5.1 to F and $C_{u,t} = \bar{C}_{u,t}$, we obtain the existence of a random time $\bar{\tau}$ associated with the Azéma submartingale F and satisfying the hypothesis (HP). In fact, $\bar{\tau}$ satisfies the hypothesis (H) since it equals (see Lemma 4.3.2)

$$\bar{\tau} = \inf\{t \in \mathbb{R}_+ : F_{t,\infty} \ge U\} = \inf\{t \in \mathbb{R}_+ : F_t \ge U\}$$

with an increasing, $\mathbb{F}$-adapted process F and thus $F_{u,u} = F_{u,t} = F_{u,\infty}$ for all $0 \le u < t$. On the other hand, the Doob-Meyer decomposition of F reads

$$F_t = -M_t + A_t = (1-\epsilon)(H^1_t - \lambda(T_1 \wedge t)) + \epsilon + \lambda(1-\epsilon)(T_1 \wedge t). \tag{4.38}$$

Since A is continuous and ${}^pM_t = M_{t-}$ (see Theorem 4.5 in Nikeghbali [91]), we deduce from (4.38) that ${}^pF_t = F_{t-}$ for all $t \in \mathbb{R}_+$. Hence, from Theorem 4.4.1, the unique predictable multiplicative system associated with F equals, for all $u \le t$ (see equation (4.19))

$$\widehat{C}_{u,t} := \exp\bigg(- \int_{(u\wedge T_1, t\wedge T_1]} \frac{\lambda(1-\epsilon)}{\epsilon + (1-\epsilon)\mathbb{1}_{\{T_1 < s\}}}\, ds \bigg).$$

By applying Theorem 4.5.1 to $\widehat{C}_{u,t}$, we obtain the existence of a random time τ such that $(\mathbb{P},\mathbb{F})$-conditional distribution of τ satisfies $\widehat{F}_{u,t} = \widehat{C}_{u,t}F_t$ for all $u \le t$. We observe that τ satisfies the hypothesis (HP), but the hypothesis (H) fails to hold. Finally, the dynamics of the $(\mathbb{P},\mathbb{F})$-conditional distributions of τ and $\bar{\tau}$ for a fixed u and $t \ge u$ differ since the dynamics of $\widehat{F}_{u,t}$ are driven by the martingale M (see (4.37)), whereas the process $\bar{F}_{u,t} = F_u$ is constant for $t \ge u$.

Remark 4.5.2. Observe that $\bar{\tau}$ is a *pseudo-stopping time* (since the hypothesis (H) holds for $\bar{\tau}$) and thus $\bar{M}$ is necessarily a constant martingale (see Theorem 1 in Nikeghbali and Yor [92]). A similar argument can be used to show that the construction based on Theorem 4.5.1 with an optional multiplicative system $\bar{C}_{u,t}$ is not capable of producing non-trivial pseudo-stopping times, that is, pseudo-stopping times for which the hypothesis (H) fails to hold.

4.5.3 Honest Times

Let us now make a few comments regarding an important class of random times extending the class of $\mathbb{F}$-stopping times, namely, the *honest times*. They were studied, for instance, by Jeulin and Yor [65, 66], Yor [107], Nikeghbali and Yor [93] and Kardaras [68].

Definition 4.5.1. A positive random variable τ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called an *honest time* if, for every $t > 0$, there exists an $\mathcal{F}_t$-measurable random variable τ_t such that τ is equal to τ_t on the event $\{\tau \le t\}$, that is, $\tau \mathbb{1}_{\{\tau \le t\}} = \tau_t \mathbb{1}_{\{\tau \le t\}}$.

An honest time can be alternatively defined as the end of an optional set. The hypothesis (H) fails to hold for an honest time, unless it is an $\mathbb{F}$-stopping time, since an honest time is an $\mathcal{F}_\infty$-measurable random variable. The following result shows, in particular, that the class of random times satisfying the hypothesis (HP) encompasses honest times.

Lemma 4.5.2. *Let τ be an honest time τ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Then there exists a multiplicative system $(C_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ associated with the Azéma submartingale $F_t = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t)$, such that $C_{t,\infty} = \mathbb{1}_{\{\tau\le t\}}$, that is, the increasing process $C_{t,\infty}$ takes only the values 0 and 1. The $(\mathbb{P}, \mathbb{F})$-conditional distribution of τ satisfies the hypothesis (HP).*

Proof. We set $C_{u,t} = \mathbb{1}_{\{\tau_t\le u\}}$. We check now that $C_{u,t} := \mathbb{1}_{\{\tau_t\le u\}}$ is a multiplicative system associated with F. It is easy to see that $C_{u,t}$ is increasing and right-continuous in u and is $\mathbb{F}$-adapted and decreasing in t. It is clear that the equality $\tau_u = \tau_t$ holds on $\{\tau_t \le u\}$ and thus $C_{u,t}$ is multiplicative since, for all $0 \le u < s < t$,

$$C_{u,s}C_{s,t} = \mathbb{1}_{\{\tau_s\le u\}}\mathbb{1}_{\{\tau_t\le s\}} = \mathbb{1}_{\{\tau_t\le u\}} = C_{u,t}.$$

By setting $t = \infty$, we obtain $C_{u,\infty} = \mathbb{1}_{\{\tau\le u\}}$. Finally, condition (4.16) holds trivially, since for all $u \ge 0$

$$\mathbb{E}_{\mathbb{P}}\left(C_{u,\infty} \,\middle|\, \mathcal{F}_u\right) = \mathbb{P}(\tau \le u \,|\, \mathcal{F}_u) = F_u.$$

We conclude that $C_{u,t} = \mathbb{1}_{\{\tau_t\le u\}}$ is a multiplicative system associated with F. Furthermore, the $(\mathbb{P}, \mathbb{F})$-conditional distribution of τ satisfies $F_{u,t} = \mathbb{E}_{\mathbb{P}}(F_u \,|\, \mathcal{F}_t)$ for $u \ge t$ and $F_{u,t} = F_t\mathbb{1}_{\{\tau_t\le u\}} = F_tC_{u,t}$ for $u < t$. Hence the hypothesis (*HP*) holds by Lemma 4.2.2 (or by Lemma 4.5.1). □

Let us now consider an arbitrary multiplicative system $C_{u,t}$. For every $t > 0$, let L_t be an $\mathcal{F}_t$-measurable random time given by

$$L_t = \sup\{0 < u \le t : C_{u,t} = 0\} = \sup\{0 < u \le t : C_{u-,t} = 0\} \tag{4.39}$$

where, by convention, $\sup\emptyset = 0$. Then $L_t \le L_s$ for $t \le s$. In particular, $L_t \le L_\infty$ where

$$L_\infty = \sup\{u > 0 : C_{u,\infty} = 0\}. \tag{4.40}$$

Since $C_{u,t}$ is a multiplicative system, it is easy to see that $C_{s,t} = 0$ for every $s \in [0, L_t)$. The following lemma is easy to prove.

Lemma 4.5.3. *The following properties are valid:*
(i) the increasing process $(L_t)_{t\in\mathbb{R}_+}$ is $\mathbb{F}$-adapted,
(ii) for every $t > 0$, the equality $L_\infty = L_t$ holds on $\{L_\infty < t\}$ so that L_∞ is an honest time.

Example 4.5.2. Consider Lemma 4.5.1 with a multiplicative system $C_{u,t}$ associated with F. Assume that the increasing process $C_{t,\infty}$ takes only values 0 and 1. Then the random time τ introduced in Lemma 4.3.1 equals $\tau = \inf\{u \in \mathbb{R}_+ : C_{u,\infty} = 1\}$ and thus $\tau = L_\infty$ where L_∞ is defined in (4.40). Hence, by part (ii) in Lemma 4.5.3, τ is an honest time.

The following result characterizes honest times as $\mathcal{F}_\infty$-measurable random times satisfying the hypothesis (*HP*).

Proposition 4.5.2. *Let τ be a random time on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Then τ is an honest time if and only if τ is $\mathcal{F}_\infty$-measurable and satisfies the hypothesis (HP).*

Proof. The 'only if' part follows from Definition 4.5.1 and Lemma 4.5.2. To prove the 'if' part, let us assume that τ is $\mathcal{F}_\infty$-measurable and satisfies the hypothesis (*HP*). Then, by Lemma 4.2.2, the random field $(C_{u,t})_{u\in\bar{\mathbb{R}}_+,\, t\ge u}$, given by $C_{u,t} := F_{u,t}F_t^{-1}$, is a multiplicative system. Moreover, by part (ii) in Lemma 4.5.3, the random time L_∞ given by (4.40) is an honest time. The assumption that the random time τ is $\mathcal{F}_\infty$-measurable implies that $C_{u,\infty} = \mathbb{1}_{\{\tau\le u\}}$ and thus $\tau = L_\infty$. Hence τ is an honest time. □

From the proof of Lemma 4.5.2, the $(\mathbb{P}, \mathbb{F})$-conditional distribution of the end of an optional set can be expressed in terms of its Azéma submartingale F and the family τ_t of $\mathcal{F}_t$-measurable random variables. This is not a fully satisfactory characterization of $F_{u,t}$ when we address the problem of finding a random time associated with F, since the random times τ_t are not given a priori. A stronger result is known to hold for ends of predictable sets: if τ is the end of an $\mathbb{F}$-predictable set then the $(\mathbb{P},\mathbb{F})$-conditional distribution is given in terms of F only; specifically, $F_{u,t}=\mathbb{E}_{\mathbb{P}}(F_u\,|\,\mathcal{F}_t)$ for $t\le u$ and $F_{u,t}=F_t\mathbb{1}_{\{\widehat{\tau}_t\le u\}}$ for $t>u$ where $\widehat{\tau}_t:=\sup\{0<u\le t:\,F_{u-}=0\}$ (see Meyer [83]). Of course, the random field $F_{u,t}$ satisfies the hypothesis (*HP*). Using Lemma 4.3.2, one may thus construct a random time τ on the extended probability space consistent with $F_{u,t}$. In view of Proposition 4.5.2, the random time τ obtained in this way is an honest time, since it satisfies the hypothesis (*HP*) and is $\mathcal{F}_\infty$-measurable (the latter property is true since $F_{t,\infty}$ takes only values 0 and 1).

The natural question is then to characterize the class of all Azéma submartingales corresponding to honest times. It is common in the literature to work under the assumption that all $\mathbb{F}$-martingales are continuous, so that any optional set is predictable. Under this postulate, the $(\mathbb{P},\mathbb{F})$-conditional distribution of an honest time is uniquely determined by its Azéma submartingale. In particular, if $F_{t-}>0$ for every $t>0$ then an honest time τ satisfies (H) and thus it is a stopping time. The interested reader is referred to Nikeghbali and Yor [93] and Kardaras [68] for a detailed study of honest times that avoid all $\mathbb{F}$-stopping times and their Azéma submartingales and their link to maxima of local martingales.

4.5.4 Non-Multiplicative Approach to Conditional Distributions

Before giving an example of a non-multiplicative approach to conditional distributions, we will first provide another necessary and sufficient condition for a $(\mathbb{P},\mathbb{F})$-conditional distribution to satisfy the hypothesis (*HP*). For simplicity, we only deal here with continuous and strictly positive $(\mathbb{P},\mathbb{F})$-conditional distributions $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$.

Let us fix $u\in\mathbb{R}_+$. We denote by $\mathcal{S}^u$ the space of all real-valued semimartingales on $[u,\infty)$ and we write

$$\mathcal{S}^u_1=\big\{U\in\mathcal{S}^u\mid\{UU_-=0\}\text{ is an evanescent set}\big\}.$$

The *stochastic logarithm* $\mathcal{L}(U)$ of $U\in\mathcal{S}^u_1$ is defined by the formula, for $t\in[u,\infty)$ (see, for instance, Section 9.4.3 in Jeanblanc et al. [61])

$$\mathcal{L}(U)_t=\int_{(u,t]}\frac{1}{U_{s-}}\,dU_s. \tag{4.41}$$

Recall that for every $U\in\mathcal{S}^u_1$ the equality $U_t=U_s\,\mathcal{E}_{s,t}(\mathcal{L}(U))$ holds for any $u\le s\le t$, where we denote $(\mathcal{E}_{s,t}(X))_{t\in[s,\infty)}$ is the stochastic exponential of X started at s. For any fixed $u\in\mathbb{R}_+$, we define the *generating martingale* $M_{u,\cdot}=(M_{u,t})_{t\in[u,\infty)}$ by setting

$$M_{u,t}:=\mathcal{L}(F_{u,\cdot})_t=\int_{(u,t]}\frac{1}{F_{u,s-}}\,dF_{u,s}.$$

Proposition 4.5.3. *Assume that the* $(\mathbb{P},\mathbb{F})$*-conditional distribution* $(F_{u,t})_{u,t\in\mathbb{R}_+}$ *is such that, for every* $u>0$*, the process* $(F_{u,t})_{t\ge u}$ *is a strictly positive martingale. Then the following statements are true:*
(i) The random field $(F_{u,t})_{u,t\in\mathbb{R}_+}$ *satisfies the hypothesis* (*HP*) *if and only if, for any fixed* $u\in\mathbb{R}_+$*,*

$$\mathcal{E}_{s,t}(M_{s,\cdot})=\mathcal{E}_{s,t}(M_{u,\cdot}),\quad\forall\,u\le s\le t. \tag{4.42}$$

(ii) Assume, in addition, that for any fixed $u\in\mathbb{R}_+$ *the process* $(F_{u,t})_{t\ge u}$ *is continuous. Then* $F_{u,t}$ *satisfies the hypothesis* (*HP*) *if and only if, for every* $u\in\mathbb{R}_+$ *and* $s\ge u$*,*

$$M_{u,t}-M_{u,s}=M_{s,t}-M_{s,s},\quad\forall\,t\ge s. \tag{4.43}$$

Proof. For part (i), we start by noting that the process $(F_{u,t})_{t\ge u}$ satisfies $F_{u,t}=F_{u,u}\,\mathcal{E}_{u,t}(M_{u,\cdot})$ where we denote $M_{u,\cdot}=\mathcal{L}(F_{u,\cdot})$. Therefore, the random field $F_{u,t}$ satisfies the hypothesis (*HP*) whenever the following equalities hold, for any fixed $u\in\mathbb{R}_+$ and all $u\le s\le t$,

$$F_{u,u}\,\mathcal{E}_{u,s}(M_{u,\cdot})\,F_{s,s}\,\mathcal{E}_{s,t}(M_{s,\cdot})=F_{u,s}F_{s,t}=F_{s,s}F_{u,t}=F_{s,s}F_{u,u}\,\mathcal{E}_{u,t}(M_{u,\cdot}).$$

Since $\mathcal{E}_{u,t}(M_{u,\cdot}) = \mathcal{E}_{u,s}(M_{u,\cdot})\mathcal{E}_{s,t}(M_{u,\cdot})$, this is equivalent to (4.42). To establish (ii), we observe that if, for every $u \in \mathbb{R}_+$, the $(\mathbb{P},\mathbb{F})$-martingale $(F_{u,t})_{t\geq u}$ is continuous then (4.42) becomes

$$\exp\Big(\int_s^t dM_{s,v} - \frac{1}{2}\int_s^t d\langle M_{s,\cdot}\rangle_v\Big) = \exp\Big(\int_s^t dM_{u,v} - \frac{1}{2}\int_s^t d\langle M_{u,\cdot}\rangle_v\Big).$$

Therefore, equality (4.42) holds whenever, for any fixed $u \in \mathbb{R}_+$ and every $u \leq s \leq t$,

$$(M_{u,t} - M_{s,t}) - (M_{u,s} - M_{s,s}) = \frac{1}{2}\left(\langle M_{u,\cdot}\rangle_t - \langle M_{s,\cdot}\rangle_t\right) - \frac{1}{2}\left(\langle M_{u,\cdot}\rangle_s - \langle M_{s,\cdot}\rangle_s\right). \tag{4.44}$$

Let us fix $u \in \mathbb{R}_+$ and $s \geq u$. Equality (4.44) means the continuous martingale $(M_{u,t} - M_{s,t})_{t\geq s}$ is a process of finite variation and thus it is constant. We conclude that $F_{u,t}$ satisfies the hypothesis (HP) whenever condition (4.43) is met. □

Remark 4.5.3. It is also clear that condition (4.44) (or (4.43)) is met whenever the dynamics of the generating martingale $(M_{u,t})_{t\geq u}$ are independent of the initial date u.

As was pointed out in Remark 4.3.1 (see also Example 4.5.1), the Azéma submartingale of a random time does not uniquely characterize its $(\mathbb{P},\mathbb{F})$-conditional distribution. We will now provide another explicit example of the said non-uniqueness. Unlike in Example 4.5.1, we will now construct a $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time failing to satisfy the hypothesis (HP). More sophisticated examples of non-uniqueness of the $(\mathbb{P},\mathbb{F})$-conditional distribution are provided in Jeanblanc and Song [60].

In Lemma 4.5.4, we consider a special case of a $(\mathbb{P},\mathbb{F})$-conditional distribution examined previously in the literature (see, for instance, Proposition 4.9 in El Karoui et al. [36]). Let Z be a positive, continuous, square-integrable $(\mathbb{P},\mathbb{F})$-martingale given on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ such that $\langle Z\rangle$ is strictly increasing (to avoid trivial cases). The following result is not hard to establish. In particular, one can check that the dynamics of the generating martingale $(M_{u,t})_{t\geq u}$ depend on u, so that (4.43) is not satisfied.

Lemma 4.5.4. *Let $\widetilde{F}_{u,t}$ be given by the expression, for all $0 \leq u \leq t$,*

$$\widetilde{F}_{u,t} = 1 - \exp\Big(-uZ_t - \frac{1}{2}u^2\langle Z\rangle_t\Big) \tag{4.45}$$

and $\widetilde{F}_{\infty,t} = 1$ for every $t \in \bar{\mathbb{R}}_+$. Then $(\widetilde{F}_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is a $(\mathbb{P},\mathbb{F})$-conditional distribution and, for a fixed $u \in \mathbb{R}_+$, the process $(\widetilde{F}_{u,t})_{u\leq t}$ satisfies

$$\widetilde{F}_{u,t} = \widetilde{F}_{u,u} + \int_u^t u(1 - \widetilde{F}_{u,s})\,dZ_s. \tag{4.46}$$

The $(\mathbb{P},\mathbb{F})$-conditional distribution $\widetilde{F}_{u,t}$ fails to satisfy the hypothesis (HP).

Our goal is now to apply the multiplicative approach to the $(\mathbb{P},\mathbb{F})$-submartingale $F = 1 - G$ where

$$G_t := \widetilde{G}_{t,t} = \exp\Big(-tZ_t - \frac{1}{2}t^2\langle Z\rangle_t\Big).$$

The Itô formula yields

$$dG_t = -tG_t\,dZ_t - G_t(Z_t + t\langle Z\rangle_t)\,dt$$

and thus the multiplicative decomposition $G_t = N_t e^{-\Lambda_t}$ of the strictly positive $(\mathbb{P},\mathbb{F})$-supermartingale G is given by

$$N_t = \exp\Big(-\int_0^t s\,dZ_s - \frac{1}{2}\int_0^t s^2\,d\langle Z\rangle_s\Big)$$

and

$$\Lambda_t = \int_0^t (Z_s + s\langle Z\rangle_s)\,ds.$$

It is also worth noting that

$$dF_t = -e^{-\Lambda_t}\, dN_t + N_t\, de^{-\Lambda_t}. \tag{4.47}$$

Let us consider the $(\mathbb{P},\mathbb{F})$-conditional distribution $(\widehat{F}_{u,t})_{u,t\in\mathbb{R}_+}$ obtained using formula (4.32) for some predictable multiplicative system $\widehat{C}_{u,t}$ associated with F. In fact, since F satisfies Assumption 4.5.1, the unique predictable multiplicative system $(\widehat{F}_{u,t})_{u,t\in\mathbb{R}_+}$ associated with F is given by (4.35). In view of Lemma 4.5.1, the $(\mathbb{P},\mathbb{F})$-conditional distribution $(\widehat{F}_{u,t})_{u\in\mathbb{R}_+,t\geq u}$ satisfies (HP).

Lemma 4.5.5. *For any fixed $u \geq 0$, the dynamics of the $(\mathbb{P},\mathbb{F})$-martingale $(\widehat{F}_{u,t})_{u\in\mathbb{R}_+,t\geq u}$ are*

$$\widehat{F}_{u,t} = \widehat{F}_{u,u} + \int_u^t \frac{\widehat{F}_{u,s}}{F_s}\, s(1-F_s)\, dZ_s. \tag{4.48}$$

Proof. Let us denote $\widehat{F}_{u,t} = \widehat{C}_{u,t}F_t$ for all $0 \leq u \leq t$. The process F is continuous and, for any $u \in \mathbb{R}_+$, the process $(\widehat{C}_{u,t})_{t\geq u}$ is continuous and decreasing. Hence the Itô integration by parts formula yields

$$d\widehat{F}_{u,t} = F_t\, d\widehat{C}_{u,t} = F_t\, d\widehat{C}_{u,t} + \widehat{C}_{u,t}\, dF_t. \tag{4.49}$$

From (4.47), we see that the martingale part in the Doob-Meyer decomposition of F equals

$$M_t = -\int_0^t e^{-\Lambda_s}\, dN_s = \int_0^t sN_s e^{-\Lambda_s}\, dZ_s = \int_0^t s(1-F_s)\, dZ_s,$$

where the second equality holds since $dN_s = -sN_s\, dZ_s$. Since we know that $(\widehat{F}_{u,t})_{t\geq u}$ is a $(\mathbb{P},\mathbb{F})$-martingale starting at time $u \geq 0$, we deduce from (4.49) (or, equivalently, from (4.21)) that

$$\widehat{F}_{u,t} = \widehat{F}_{u,u} + \int_u^t \widehat{C}_{u,s}\, dM_s = F_u + \int_u^t \frac{\widehat{F}_{u,s}}{F_s}\, dM_s = F_u + \int_u^t \frac{\widehat{F}_{u,s}}{F_s}\, s(1-F_s)\, dZ_s,$$

which is the desired formula. □

It is clear that, for a fixed $u \in \mathbb{R}_+$, the processes $(\widetilde{F}_{u,t})_{t\geq u}$ and $(\widehat{F}_{u,t})_{t\geq u}$ have different dynamics. Using Lemma 4.3.2, one may construct two random times associated with F, but with different $(\mathbb{P},\mathbb{F})$-conditional distributions; one of them satisfies the hypothesis (HP), whereas the other does not.

Chapter 5

Progressive Enlargement of Filtration with Pseudo-Honest Times

This chapter is based on the paper with the same title by Li and Rutkowski [75] submitted to the *Annals of Applied Probability* in March 2012.

5.1 Introduction

We continue here the research from Li and Rutkowski [74] (Chapter 4), by addressing the issues related to properties of enlarged filtrations for various classes of random times. We work throughout on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$ satisfying the usual conditions. It is assumed throughout that τ is any $\mathbb{R}_+$-valued random time given on this space.

Definition 5.1.1. An *enlargement of* $\mathbb{F}$ *associated with* τ is any filtration $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$ in $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the usual conditions, and such that: (i) the inclusion $\mathbb{F} \subset \mathbb{K}$ holds, meaning that $\mathcal{F}_t \subset \mathcal{K}_t$ for all $t \in \mathbb{R}_+$, and (ii) τ is a $\mathbb{K}$-stopping time.

Let us recall two particular enlargements, which were extensively studied in the literature (see, e.g., Dellacherie and Meyer [32], Jacod [50], Jeanblanc and Le Cam [57], Jeulin [62, 63, 64], Jeulin and Yor [65, 66, 67], and Yor [107, 108]). Section 8 in Nikeghbali [91] provides an overview of the most pertinent results, but mostly under either the postulate (**C**) that all $\mathbb{F}$-martingales are continuous and/or the postulate (**A**) that the random time τ avoids all $\mathbb{F}$-stopping times.

Definition 5.1.2. The *initial enlargement* of $\mathbb{F}$ is the filtration $\mathbb{G}^* = (\mathcal{G}^*_t)_{t\geq 0}$ where the σ-field $\mathcal{G}^*_t$ given by the equality $\mathcal{G}^*_t = \cap_{s>t} (\sigma(\tau) \vee \mathcal{F}_s)$ for all $t \in \mathbb{R}_+$.

The initial enlargement does not seem to be well suited for a general analysis of properties of a random time with respect to a reference filtration $\mathbb{F}$ since it implies, in particular, that $\sigma(\tau) \subset \mathcal{G}^*_0$, meaning that all the information about τ is already available at time 0 (note, however, that this feature can indeed be justified when dealing with some problems related to the so-called *insider trading*). One can argue that the following notion of the progressive enlargement of $\mathbb{F}$ with observations of a random time is more suitable for formulating and solving various problems associated with an additional information conveyed by a random time τ.

Definition 5.1.3. The *progressive enlargement* of $\mathbb{F}$ is the minimal enlargement, that is, the smallest filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, satisfying the usual conditions, such that $\mathbb{F} \subset \mathbb{G}$ and τ is a $\mathbb{G}$-stopping time. More explicitly, $\mathcal{G}_t = \cap_{s>t} \mathcal{G}^o_s$ where we denote $\mathcal{G}^o_t = \sigma(\tau \wedge t) \vee \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

Let $\mathbb{H}$ be the filtration generated by the indicator process $H_t = \mathbb{1}_{\{\tau \le t\}}$. It is apparent that the inclusions $\mathbb{F} \vee \mathbb{H} \subset \mathbb{G} \subset \mathbb{G}^*$ are valid and, in fact, $\mathbb{G}$ coincides with the minimal enlargement of $\mathbb{F} \vee \mathbb{H}$ satisfying the usual conditions. In what follows, we will mainly work with the progressive enlargement $\mathbb{G}$, although in some circumstances we will also make use of the initial enlargement $\mathbb{G}^*$.

Recall that for any two filtrations $\mathbb{F} \subset \mathbb{K}$ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, the *hypothesis* (H') holds for $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$ whenever any $(\mathbb{P}, \mathbb{F})$-semimartingale is also a $(\mathbb{P}, \mathbb{K})$-semimartingale (see, e.g., Dellacherie and Meyer [32], Jeulin [63], Jeulin and Yor [66] or Yor [107]). The problem of checking whether the hypothesis (H') is satisfied and finding the canonical semimartingale decomposition of a $(\mathbb{P}, \mathbb{F})$-special semimartingale with respect to a progressive enlargement $\mathbb{G}$ of a filtration $\mathbb{F}$ have attracted a considerable attention and were examined in several papers during the past thirty years. In particular, the following fundamental properties are worth to be recalled:
(i) a $(\mathbb{P}, \mathbb{F})$-semimartingale may fail to be a $(\mathbb{P}, \mathbb{G})$-semimartingale, in general,
(ii) any $(\mathbb{P}, \mathbb{F})$-special semimartingale stopped at τ is a $(\mathbb{P}, \mathbb{G})$-special semimartingale,
(iii) any $(\mathbb{P}, \mathbb{F})$-special semimartingale is a $(\mathbb{P}, \mathbb{G})$-special semimartingale when τ is an *honest time* with respect to a filtration $\mathbb{F}$, that is, the random variable τ is an end of some $\mathbb{F}$-optional set.

Furthermore, by the classic result due to Jacod [50], the hypothesis (H') is satisfied in the case of the initial enlargement of $\mathbb{F}$ provided that τ is an *initial time*. Recall that τ is called an *initial time* with respect to a filtration $\mathbb{F}$ if there exists a measure η on $(\bar{\mathbb{R}}_+, \mathcal{B}(\bar{\mathbb{R}}_+))$ such that the $(\mathbb{P}, \mathbb{F})$-conditional distributions of τ are absolutely continuous with respect to η, that is, $F_{du,t} \ll \eta(du)$. This property is also frequently referred to as the *density hypothesis*. In the path-breaking paper by Jeulin and Yor [65] (see also Jeulin and Yor [66]), the authors derived the $(\mathbb{P}, \mathbb{G})$-semimartingale decomposition of the stopped process $U_{\tau \wedge t}$ for any random time τ and any $(\mathbb{P}, \mathbb{F})$-local martingale U. They also obtained the $(\mathbb{P}, \mathbb{G})$-semimartingale decomposition of an arbitrary $(\mathbb{P}, \mathbb{F})$-local martingale U under an additional assumption that τ is an honest time with respect to the filtration $\mathbb{F}$. The latter result was recently extended to the case of initial times by El Karoui et al. [36], Jeanblanc and Le Cam [57], Kchia et al. [70] and Nikeghbali and Yor [93]. For obvious reasons, we are not in a position to discuss all abovementioned papers in detail here, although some results from them will be quoted or referred to in what follows. Let us only mention here that, by the theorem due to Stricker [104], for arbitrary two filtrations $\mathbb{F} \subset \mathbb{K}$, any a $(\mathbb{P}, \mathbb{K})$-semimartingale which is also an $\mathbb{F}$-adapted process is necessarily a $(\mathbb{P}, \mathbb{F})$-semimartingale. Therefore, in order to prove that the hypothesis (H') holds for a filtration $\mathbb{F}$ and its progressive enlargement $\mathbb{G}$, it suffices to show that the hypothesis (H') is satisfied by $\mathbb{F}$ and any filtration $\mathbb{K}$ such that $\mathbb{G} \subset \mathbb{K}$. A typical choice of $\mathbb{K}$ in this context is the initial enlargement $\mathbb{G}^*$.

The hypothesis (H') should be contrasted with the stronger *hypothesis* (H) for $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$, which is also frequently referred to as the *immersion property* between $\mathbb{F}$ and $\mathbb{K}$. This hypothesis, which stipulates that any $(\mathbb{P}, \mathbb{F})$-local martingale is also a $(\mathbb{P}, \mathbb{K})$-local martingale, was first studied in the paper by Brémaud and Yor [20]. In the case of the progressive enlargement $\mathbb{G}$ of a filtration $\mathbb{F}$ through a random time τ defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$, the immersion property for $\mathbb{F}$ and $\mathbb{G}$ is well known to be equivalent to the hypothesis (H) introduced in Definition 5.2.2 below (see, e.g., Elliott et al. [36]) and thus no confusion may arise. Note also that the hypothesis (H), unlike the hypothesis (H'), is not invariant under an equivalent change of a probability measure. However, as shown in Jeulin and Yor [66] (Proposition 2), if the hypothesis (H) is satisfied under $\mathbb{P}$ by $\mathbb{F}$ and an arbitrary enlargement $\mathbb{K}$ and a probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}$ then the hypothesis (H') necessarily holds for $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{Q}$.

The main hypotheses examined in the present work are the *hypothesis* (HP) and the *extended density hypothesis* (the *hypothesis* (ED), for short), as specified in Definitions 5.2.2 and 5.2.5, respectively. The corresponding classes of random times are termed *pseudo-honest times* and *pseudo-initial times*. The hypothesis (HP) is clearly weaker than the hypothesis (H) and it is known to hold, in particular, when a random time is constructed using the *multiplicative approach* (see Li and Rutkowski [74]), as well as for the alternative construction of a random time developed in Jeanblanc and Song [59]. It was also shown in [74] that, under mild technical assumption, the hypothesis (HP) is equivalent to the *separability* of the $(\mathbb{P}, \mathbb{F})$-conditional distribution of τ (see Definition 5.2.4). The hypothesis (ED) extends the density hypothesis; it is introduced in order to avoid the awkward assumption on strict positivity of $(\mathbb{P},\mathbb{F})$-conditional distribution of the random time τ. It is worth to point out that most results obtained for initial times can be extended to this new setting.

This chapter is organized as follows. In Section 5.2, we recall some basic properties of $(\mathbb{P},\mathbb{F})$-conditional distributions of random times and enlarged filtrations. In particular, we provide an alternative characterization of the progressive enlargement $\mathbb{G}$. This alternative characterization of $\mathbb{G}$ is used in Section 5.3 in computations of conditional expectation of $\mathbb{G}$-adapted processes under the hypothesis (HP) and the extended density hypothesis. Subsequently, in Section 5.4, we provide sufficient conditions for a $\mathbb{G}$-adapted process to be a $(\mathbb{P},\mathbb{G})$-martingale. Explicit computations of the $\mathbb{G}$-compensator (that is, the $(\mathbb{P},\mathbb{G})$-dual predictable projection) of the indicator process $H_t = \mathbb{1}_{\{\tau\leq t\}}$ are provided in Section 5.5. Main results of this chapter are established in Section 5.6 in which the validity of the hypothesis (H') is studied for the progressive enlargement of the underlying filtration $\mathbb{F}$ through either a pseudo-honest or a pseudo-initial random time. We extend there several related results from the existing literature. First, in Theorem 5.6.2, we compute a general semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-martingale with respect to the progressively enlarged filtration $\mathbb{G}$ when τ is assumed to be a pseudo-honest time. Particular examples of this decomposition are subsequently examined in 5.6.1 in which we postulate that a random time was constructed using the multiplicative approach developed in [74] using either a predictable or an optional multiplicative system associated with a given in advance Azéma submartingale F. Finally, in Section 5.6.2, we deals with the corresponding results for a pseudo-initial time. It is worth stressing that results on a $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-local martingale are crucial for applications in financial mathematics, especially in credit risk models, where a random time τ represents the moment of occurrence of some credit event (e.g., a default). Two examples of applications of our results to problems of financial mathematics are outlined in Section 5.7.

5.2 Random Times and Filtrations

In this section, we deal with the most pertinent properties of random times and the associated enlargements of a reference filtration $\mathbb{F}$. For more details, we refer to [74] where, in particular, various constructions of a random time are examined. The interested reader may also consult papers by Jeanblanc and Song [59, 60] for closely related results.

5.2.1 Properties of Conditional Distributions

Let us first introduce the notation for several pertinent characteristics of a finite random time τ defined on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. The $(\mathbb{P},\mathbb{F})$-supermartingale $G_t = \mathbb{P}(\tau > t\,|\,\mathcal{F}_t)$ is commonly known as the *Azéma supermartingale* of τ. We will sometimes refer to the $(\mathbb{P},\mathbb{F})$-submartingale $F = 1 - G$ as the *Azéma submartingale* of τ. The $(\mathbb{P},\mathbb{F})$-*conditional distribution of* τ is the random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ given by

$$F_{u,t} = \mathbb{P}(\tau \leq u\,|\,\mathcal{F}_t), \quad \forall\, u,t \in \bar{\mathbb{R}}_+. \tag{5.1}$$

The following definition characterizes the class of all conditional distributions of a random time.

Definition 5.2.1. A random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ is said to be an $(\mathbb{P},\mathbb{F})$-*conditional distribution* if it satisfies:
(i) for every $u \in \bar{\mathbb{R}}_+$ and $t \in \bar{\mathbb{R}}_+$, we have $0 \leq F_{u,t} \leq 1$, $\mathbb{P}$-a.s.,
(ii) for every $u \in \bar{\mathbb{R}}_+$, the process $(F_{u,t})_{t\in\bar{\mathbb{R}}_+}$ is a $(\mathbb{P},\mathbb{F})$-martingale,
(iii) for every $t \in \bar{\mathbb{R}}_+$, the process $(F_{u,t})_{u\in\bar{\mathbb{R}}_+}$ is right-continuous, increasing and $F_{\infty,t} = 1$.

Note that for every $u \in \bar{\mathbb{R}}_+$, conditions (i)-(ii) in Definition 5.2.1 imply that $F_{u,\infty} = \lim_{t\to\infty} F_{u,t}$ and $F_{u,t} = \mathbb{E}_{\mathbb{P}}(F_{u,\infty}\,|\,\mathcal{F}_t)$ for every $t \in \bar{\mathbb{R}}_+$. Since (iii) yields $F_{u,t} \leq F_{s,t}$ for all $u \leq s$, the (non-adapted) process $(F_{u,\infty})_{u\in\bar{\mathbb{R}}_+}$ is increasing and thus it admits a càdlàg version. It is known that for any random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ there exists a random time τ on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ such that (5.1) holds.

Let us examine some pertinent properties of conditional distributions of random times. Throughout this section, by a $(\mathbb{P},\mathbb{F})$-conditional distribution, we mean any random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ satisfying Definition 5.2.1. We first recall the classic hypothesis (H), which was studied in numerous papers (see, e.g., Brémaud and Yor [20] or Elliott et al. [35]), and its generalization termed the hypothesis (HP) (it is obvious that the hypothesis (H) implies (HP)).

Definition 5.2.2. A $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is said to satisfy:
(i) the *hypothesis* (H) whenever for all $0\leq u\leq s<t$

$$F_{u,s}=F_{u,t}, \tag{5.2}$$

(ii) the *hypothesis* (HP) whenever for all $0\leq u<s<t$

$$F_{u,s}F_{s,t}=F_{s,s}F_{u,t}. \tag{5.3}$$

It was shown in [74] that any honest time satisfies the hypothesis (HP) and, in fact, an $\mathcal{F}_\infty$-measurable random time τ is an honest time if and only if it satisfies the hypothesis (HP). This motivates us to say that a random time is a *pseudo-honest time* with respect to $\mathbb{F}$ whenever the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ satisfies the hypothesis (HP).

Remark 5.2.1. Let us observe that if $F_{u,t}$ satisfies the hypothesis (HP) then, for all $0\leq u\leq s\leq t$,

$$\frac{F_{u,s}}{F_{s,s}}\,F_{s,t}=F_{u,t}. \tag{5.4}$$

Note that the inclusion $\{F_{s,s}=0\}\subset\{F_{u,s}=0\}$ is valid for all $0\leq u\leq s$ and, by convention, $0/0=0$. More generally, the inclusion $\{F_{u,s}=0\}\subset\{F_{u,t}=0\}$ is known to hold for all $u\leq s\leq t$ (see the proof of Lemma 2.2 in [74]).

Let us recall the concept of *separability* property of a $(\mathbb{P},\mathbb{F})$-conditional distribution (see [59, 74]).

Definition 5.2.3. We say that a $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is *completely separable* if there exists a positive, $\mathbb{F}$-adapted, increasing process K and a positive $(\mathbb{P},\mathbb{F})$-martingale L such that $F_{u,t}=K_uL_t$ for every $u,t\in\mathbb{R}_+$ such that $0\leq u\leq t$.

It is easily seen that the complete separability of $F_{u,t}$ implies that the hypothesis (HP) holds. Indeed, we have that $F_{u,s}F_{s,t}=(K_uL_s)(K_sL_t)=(K_sL_s)(K_uL_t)=F_{s,s}F_{u,t}$ for all $0\leq u<s<t$. It appears, however, that the property of complete separability is too restrictive, since it does not cover all cases of our interest. This motivates the weaker concept of *separability* (termed partial separability in [59]).

Definition 5.2.4. We say that a $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is *separable at* $v\geq 0$ if there exist a positive $(\mathbb{P},\mathbb{F})$-martingale $(L^v_t)_{t\in\mathbb{R}_+}$ and a positive, $\mathbb{F}$-adapted, increasing process $(K^v_u)_{u\in[v,\infty)}$ such that the equality $F_{u,t}=K^v_uL^v_t$ holds for every $v\leq u\leq t$. A $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is called *separable* if it is separable at all $v>0$.

Remark 5.2.2. It is known that if the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is separable and $F_0=0$ then the hypothesis (HP) holds, and thus τ is a pseudo-honest time. Conversely, if τ is a pseudo-honest time and its $(\mathbb{P},\mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is non-degenerate then the random field $F_{u,t}$ is separable. For proofs of these properties and more details, the interested reader is referred to [74].

The next definition gives a natural extension of the *density hypothesis*, which was introduced by Jacod [50] and subsequently studied by numerous authors (see, e.g., [36, 57]). Since random times satisfying the density hypothesis are called *initial times*, we find it natural to say that a random time is an *pseudo-initial time* whenever it satisfies Definition 5.2.5. From [50], the hypothesis (H') is known to hold for the initial (and thus the progressive) enlargement of $\mathbb{F}$ with an initial time.

Definition 5.2.5. A $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is said to satisfy the *extended density hypothesis* (or, briefly, the *hypothesis* (*ED*)) if there exists a random field $(m_{s,t})_{s\geq 0,\, t\geq s}$ and an $\mathbb{F}$-adapted, increasing process D such that, for all $0 \leq u \leq t$,

$$F_{u,t} = \int_{[0,u]} m_{s,t}\, dD_s \tag{5.5}$$

and, for every $s \in \mathbb{R}_+$, the process $(m_{s,t})_{t\geq s}$ is a positive $(\mathbb{P},\mathbb{F})$-martingale.

As one might guess, the results obtained under the extended density hypothesis are similar to those proven under the usual density hypothesis, that is, for initial times. Nevertheless, it is convenient to introduce it here, since it will allow us to circumvent an awkward non-degeneracy condition of the $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time, which will be needed, for instance, in the proof of Proposition 5.4.1. Moreover, it is worth noting that the extended density hypothesis is also satisfied when a pseudo-honest time is constructed through the multiplicative construction, as shown in Remark 2.1 in [75] (for a special case, see also Theorem 5.2 in Jeanblanc and Song [59]). Hence the study of pseudo-initial times is related to our main goal which is to examine pseudo-honest times.

5.2.2 Enlargements of Filtrations

We will now analyze the basic properties of various enlargements of $\mathbb{F}$ associated with a random time τ. When studying semimartingale decompositions of processes stopped at τ, it is common to use, at least implicitly, the following concept, formally introduced by Guo and Zeng [47].

Definition 5.2.6. An enlargement $\mathbb{K}$ of a filtration $\mathbb{F}$ is said to be *admissible before* τ if the equality $\mathcal{K}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}$ holds for every $t \in \mathbb{R}_+$.

In the case of a general (i.e., not necessarily honest) random time, we find it convenient to introduce the following notion, stemming from a remark in Meyer [86]. Recall that the initial enlargement $\mathbb{G}^*$ was introduced in Definition 5.1.2.

Definition 5.2.7. The family $\widehat{\mathbb{G}} = (\widehat{\mathcal{G}}_t)_{t\in\mathbb{R}_+}$ is defined by setting, for all $t \in \mathbb{R}_+$,

$$\widehat{\mathcal{G}}_t = \{A \in \mathcal{G} \,|\, \exists A_t \in \mathcal{F}_t \text{ and } A_t^* \in \mathcal{G}_t^* \text{ such that } A = (A_t \cap \{\tau > t\}) \cup (A_t^* \cap \{\tau \leq t\})\}.$$

We note that, for all $t \in \mathbb{R}_+$,

$$\widehat{\mathcal{G}}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}, \quad \widehat{\mathcal{G}}_t \cap \{\tau \leq t\} = \mathcal{G}_t^* \cap \{\tau \leq t\}. \tag{5.6}$$

It can be checked that the σ-field $\widehat{\mathcal{G}}_t$ is uniquely characterized by conditions (5.6). The next elementary result shows that the family $\widehat{\mathbb{G}}$ coincides in fact with the progressive enlargement $\mathbb{G}$, which was introduced in Definition 5.1.3 (for the proof of the lemma, we refer to [75]).

Lemma 5.2.1. *For any random time τ the progressive enlargement $\mathbb{G}$ coincides with the filtration $\widehat{\mathbb{G}}$.*

Proof. Recall that $\mathcal{G}_t = \cap_{s>t}(\sigma(\tau \wedge s) \vee \mathcal{F}_s)$ and $\mathcal{G}_t^* = \cap_{s>t}(\sigma(\tau) \vee \mathcal{F}_s)$. To show that $\widehat{\mathcal{G}}_t = \mathcal{G}_t$, it suffices to check that conditions (5.6) are satisfied by $\mathcal{G}_t$. The following relationship for all $t \in \mathbb{R}_+$ is immediate

$$\mathcal{F}_t \cap \{\tau > t\} \subset \mathcal{G}_t \cap \{\tau > t\} \subset \widetilde{\mathcal{G}}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}.$$

This shows $\mathcal{G}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}$, while also

$$\mathcal{G}_t \cap \{\tau \leq t\} = \cap_{s>t}(\sigma(\tau \wedge s) \vee \mathcal{F}_s) \cap \{\tau \leq t\} = \cap_{s>t}(\sigma(\tau) \vee \mathcal{F}_s) \cap \{\tau \leq t\} = \mathcal{G}_t^* \cap \{\tau \leq t\}$$

since $\sigma(\tau \wedge s) \cap \{\tau \leq t\} = \sigma(\tau) \cap \{\tau \leq t\}$ for every $s > t$. $\square$

It is easy to see that the filtration $\widehat{\mathbb{G}}$ is admissible before τ. When dealing with a semimartingale decomposition of an $\mathbb{F}$-martingale after τ we will use the following definition.

Definition 5.2.8. We say that an enlargement $\mathbb{K}$ is *admissible after* τ if the equality $\mathcal{K}_t \cap \{\tau \leq t\} = \mathcal{G}^*_t \cap \{\tau \leq t\}$ holds for every $t \in \mathbb{R}_+$.

It is clear that the filtration $\widehat{\mathbb{G}}$ (and thus also $\mathbb{G}$) is admissible after τ for any random time. Note also that if an enlargement $\mathbb{K}$ is admissible before and after τ then necessarily $\mathbb{K} = \mathbb{G}$. For the proof of the next elementary lemma, we refer to [75].

Lemma 5.2.2. *For any integrable, $\mathcal{G}$-measurable random variable X and any enlargement $\mathbb{K} = (\mathcal{K}_t)_{t \geq 0}$ admissible after τ we have that, for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau \leq t\}} X \,\middle|\, \mathcal{K}_t\right) = \lim_{s \downarrow t} \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau \leq t\}} X \,\middle|\, \sigma(\tau) \vee \mathcal{F}_s\right). \tag{5.7}$$

Proof. It suffices to show that

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau \leq t\}} X \,\middle|\, \mathcal{K}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau \leq t\}} X \,\middle|\, \mathcal{G}^*_t\right). \tag{5.8}$$

The second equality in (5.7) will then follow from Corollary 2.4 in [97] since $\mathcal{G}^*_t = \cap_{s>t}\left(\sigma(\tau) \vee \mathcal{F}_s\right)$. To establish (5.8), we will first check that, for every $A \in \mathcal{K}_t$,

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_A \mathbb{1}_{\{\tau \leq t\}} X\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_A \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}}\left(X \,\middle|\, \mathcal{G}^*_t\right)\right).$$

Since, by assumption, $\mathcal{K}_t \cap \{\tau \leq t\} = \mathcal{G}^*_t \cap \{\tau \leq t\}$, there exists an event $B \in \mathcal{G}^*_t$ such that $A \cap \{\tau \leq t\} = B \cap \{\tau \leq t\}$. Consequently,

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_A \mathbb{1}_{\{\tau \leq t\}} X\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B \mathbb{1}_{\{\tau \leq t\}} X\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_B \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}^*_t)\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_A \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}^*_t)\big).$$

Hence

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \leq t\}} X \,|\, \mathcal{K}_t) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}^*_t) \,|\, \mathcal{K}_t\big) = \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}^*_t),$$

since the random variable $\mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}}\left(X \,\middle|\, \mathcal{G}^*_t\right)$ is $\mathcal{K}_t$-measurable. □

5.3 Conditional Expectations under Progressive Enlargements

In the rest of the chapter, we work under the assumption that the $(\mathbb{P}, \mathbb{F})$-conditional distribution of a random time τ satisfies either the hypothesis (*HP*) or the hypothesis (*ED*), which were introduced in Definitions 5.2.2 and 5.2.5, respectively. In addition, the special case of the complete separability will be examined as well. We will need the following auxiliary result, which ensures that the processes $\mathbb{1}_{\{\tau > t\}}(G_t)^{-1}$ and $\mathbb{1}_{\{\tau \leq t\}}(F_t)^{-1}$ are well defined (for its proof, see [75]).

Lemma 5.3.1. *The following inclusions hold, for every $t \in \mathbb{R}_+$: (i) $\{\tau > t\} \subset \{G_t > 0\}$, $\mathbb{P}$-a.s., and (ii) $\{\tau \leq t\} \subset \{F_t > 0\}$, $\mathbb{P}$-a.s.*

Proof. Let us denote $A = \{F_t = 1\} = \{\mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = 1\}$. Since $A \in \mathcal{F}_t$

$$\mathbb{P}(A) = \int_A F_t \, d\mathbb{P} = \int_A \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) \, d\mathbb{P} = \int_A \mathbb{1}_{\{\tau \leq t\}} \, d\mathbb{P} = \mathbb{P}(A \cap \{\tau \leq t\}).$$

Hence $A = \{F_t = 1\} = \{G_t = 0\} \subset \{\tau \leq t\}$, $\mathbb{P}$-a.s., and thus $\{\tau > t\} \subset \{G_t > 0\}$, $\mathbb{P}$-a.s. For part (ii), let us denote $B = \{G_t = 1\} = \{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t) = 1\}$. Since $B \in \mathcal{F}_t$

$$\mathbb{P}(B) = \int_B G_t \, d\mathbb{P} = \int_B \mathbb{P}(\tau > t \,|\, \mathcal{F}_t) \, d\mathbb{P} = \int_B \mathbb{1}_{\{\tau > t\}} \, d\mathbb{P} = \mathbb{P}(B \cap \{\tau > t\}).$$

Hence $B = \{G_t = 1\} = \{F_t = 0\} \subset \{\tau > t\}$, $\mathbb{P}$-a.s., and thus $\{\tau \leq t\} \subset \{F_t > 0\}$, $\mathbb{P}$-a.s. □

Remark 5.3.1. Part (i) in Lemma 5.3.1 can also be demonstrated as follows. Let $\tau_0 = \inf\{t \in \mathbb{R}_+ : G_t = 0 \text{ or } G_{t-} = 0\}$. Since G is a supermartingale, it is equal to zero after τ_0 and thus

$$\mathbb{P}(\tau_0 < \tau) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau_0 < \tau\}}) = \mathbb{E}_{\mathbb{P}}(G_{\tau_0}\mathbb{1}_{\{\tau_0 < \infty\}}) = 0.$$

This in turn implies that $\{\tau > t\} \subset \{G_t > 0\}$, $\mathbb{P}$-a.s.

Remark 5.3.2. Let us set, by convention, $0/0 = 0$. Hence, by Lemma 5.3.1, the quantities $\mathbb{1}_{\{\tau > t\}} G_t^{-1}$ and $\mathbb{1}_{\{\tau \le t\}} F_t^{-1}$ are well defined for all t, $\mathbb{P}$-a.s.

5.3.1 Conditional Expectations for Pseudo-Honest Times

For a fixed $T > 0$, we consider the map $U_T : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and we use the notation $(u, \omega) \mapsto U_{u,T}(\omega)$. We postulate that U_T is a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_T$-measurable map, so that $U_{\tau,T}$ is a $\sigma(\tau) \vee \mathcal{F}_T$-measurable random variable. The following result corresponds to Theorem 3.1 in El Karoui et al. [36], where the case of the density hypothesis was studied.

Lemma 5.3.2. *Let $U_{\cdot,T} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_T$-measurable map. Assume that τ is a pseudo-honest time and the random variable $U_{\tau,T}$ is $\mathbb{P}$-integrable. Then:*
(i) For every $t \in [0, T)$, we have that

$$\mathbb{E}_{\mathbb{P}}(U_{\tau,T} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_{t,T} + \mathbb{1}_{\{\tau \le t\}} \widehat{U}_{\tau,t,T}$$

where

$$\widetilde{U}_{t,T} = (G_t)^{-1}\, \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau > t\}} U_{\tau,T} \,|\, \mathcal{F}_t\big) = (G_t)^{-1}\, \mathbb{E}_{\mathbb{P}}\bigg(\int_{(t,\infty]} U_{v,T}\, dF_{v,T} \,\bigg|\, \mathcal{F}_t \bigg) \tag{5.9}$$

and, for all $0 \le u \le t < T$,

$$\widehat{U}_{u,t,T} = (F_t)^{-1}\, \mathbb{E}_{\mathbb{P}}(F_{t,T} U_{u,T} \,|\, \mathcal{F}_t). \tag{5.10}$$

(ii) If, in addition, $F_{u,t}$ is completely separable so that $F_{u,t} = K_u L_t$ for $u \le t$ then (5.9) yields

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{T \ge \tau > t\}} U_{\tau,T} \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{P}}\bigg(L_T \int_{(t,T]} U_{v,T}\, dK_v \,\bigg|\, \mathcal{F}_t \bigg)$$

and (5.10) becomes

$$\widehat{U}_{u,t,T} = (L_t)^{-1}\, \mathbb{E}_{\mathbb{P}}(L_T U_{u,T} \,|\, \mathcal{F}_t).$$

Proof. The derivation of (5.9) is rather standard. Note that the hypothesis (HP) is not needed here and we may take $t \in [0, T]$. It suffices to take $U_{u,T} = g(u)\mathbb{1}_A$ for a Borel measurable map $g : \mathbb{R}_+ \to \mathbb{R}$ and an event $A \in \mathcal{F}_T$ such that the random variable $U_{\tau,T} = g(\tau)\mathbb{1}_A$ is $\mathbb{P}$-integrable. Using part (i) in Lemma 5.3.1 and the well-known formula for the conditional expectation with respect to $\mathcal{G}_t$, we obtain

$$\begin{aligned}
\mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}}(U_{\tau,T} \,|\, \mathcal{G}_t) &= \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau > t\}} U_{\tau,T} \,|\, \mathcal{F}_t\big)}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} (G_t)^{-1}\, \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_A\, \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau > t\}} g(\tau) \,|\, \mathcal{F}_T\big) \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{1}_{\{\tau > t\}} (G_t)^{-1}\, \mathbb{E}_{\mathbb{P}}\bigg(\mathbb{1}_A \int_{(t,\infty]} g(v)\, dF_{v,T} \,\bigg|\, \mathcal{F}_t \bigg) = \mathbb{1}_{\{\tau > t\}} (G_t)^{-1}\, \mathbb{E}_{\mathbb{P}}\bigg(\int_{(t,\infty]} g(v) \mathbb{1}_A\, dF_{v,T} \,\bigg|\, \mathcal{F}_t \bigg) \\
&= \mathbb{1}_{\{\tau > t\}} (G_t)^{-1}\, \mathbb{E}_{\mathbb{P}}\bigg(\int_{(t,\infty]} U_{v,T}\, dF_{v,T} \,\bigg|\, \mathcal{F}_t \bigg) = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_{t,T}
\end{aligned}$$

where $\widetilde{U}_{t,T}$ is given by (5.9). We observe that on the event $\{\tau \le t\}$ any $\mathcal{G}_t$-measurable random variable can be represented by a $\sigma(\tau) \vee \mathcal{F}_t$-measurable random variable $H_{\tau,t}$. Let us take any $t \in [0, T)$. To establish (5.10), we need to evaluate $\mathbb{E}_{\mathbb{P}}\left(U_{\tau,T} \,\middle|\, \mathcal{G}_t\right)$ on the event $\{\tau \le t\}$. An application of Lemma 5.2.2 yields

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau \le t\}} U_{\tau,T} \,\middle|\, \mathcal{G}_t\right) = \lim_{s \downarrow t} \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau \le t\}} U_{\tau,T} \,\middle|\, \sigma(\tau) \vee \mathcal{F}_s\right).$$

We first compute the conditional expectation $\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\le t\}}U_{\tau,T}\,\middle|\,\sigma(\tau)\vee\mathcal{F}_s\right)$ for $0\le t<s<T$. Recall that the hypothesis (HP) means that the equality $F_{u,s}F_{s,T}=F_{s,s}F_{u,T}$ holds for all $0\le u<s<T$, which implies that $F_{s,T}\,dF_{u,s}=F_s\,dF_{u,T}$ for any fixed $s<T$ and all $u\in[0,s]$.

Hence, for any bounded, $\sigma(\tau)\vee\mathcal{F}_s$-measurable random variable $H_{\tau,s}$, we obtain

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\le t\}}H_{\tau,s}U_{\tau,T}\big) &= \mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\le t\}}H_{\tau,s}U_{\tau,T}\,|\,\mathcal{F}_T\big)\Big) = \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]}H_{u,s}U_{u,T}\,dF_{u,T}\bigg)\\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]}H_{u,s}(F_s)^{-1}F_{s,T}U_{u,T}\,dF_{u,s}\bigg) = \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]}H_{u,s}(F_s)^{-1}\mathbb{E}_{\mathbb{P}}(F_{s,T}U_{u,T}\,|\,\mathcal{F}_s)\,dF_{u,s}\bigg)\\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]}H_{u,s}\widehat{U}_{u,s,T}\,dF_{u,s}\bigg) = \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{\tau\le t\}}H_{\tau,s}\widehat{U}_{\tau,s,T}\Big),
\end{aligned}
$$

since $\{\tau\le t\}\subset\{\tau\le s\}\subset\{F_s>0\}$, $\mathbb{P}$-a.s. (see part (ii) in Lemma 5.3.1). This in turn yields

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\le t\}}U_{\tau,T}\,\middle|\,\mathcal{G}_t\right) &= \lim_{s\downarrow t}\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\le t\}}U_{\tau,T}\,\middle|\,\sigma(\tau)\vee\mathcal{F}_s\right) = \lim_{s\downarrow t}\mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,s,T}\\
&= \lim_{s\downarrow t}\mathbb{1}_{\{\tau\le t\}}(F_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\left(F_{s,T}U_{u,T}\,\middle|\,\mathcal{F}_s\right)_{u=\tau}\\
&= \mathbb{1}_{\{\tau\le t\}}(F_t)^{-1}\,\mathbb{E}_{\mathbb{P}}\left(F_{t,T}U_{u,T}\,\middle|\,\mathcal{F}_t\right)_{u=\tau} = \mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,t,T}
\end{aligned}
$$

where the penultimate equality holds by the right-continuity of the filtration $\mathbb{F}$ and the right-continuity of processes F and $F_{\cdot,T}$. This completes the proof of part (i). For part (ii), we observe that if, in addition, the random field $F_{u,t}$ is completely separable then the asserted formulae follow from equations (5.9) and (5.10). □

Remark 5.3.3. Let us observe that for $t=0$, we first obtain, on the event $\{\tau=0\}$,

$$
\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau=0\}}H_{\tau,s}U_{\tau,T}\big) = \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{\tau=0\}}H_{\tau,s}\widehat{U}_{\tau,s,T}\Big) = \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{\tau=0\}}H_{0,s}\widehat{U}_{0,s,T}\Big)
$$

where, on the event $\{\tau=0\}\subset\{\tau\le s\}\subset\{F_s>0\}$,

$$
\widehat{U}_{0,s,T} = (F_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(F_{s,T}U_{0,T}\,|\,\mathcal{F}_s).
$$

In the second step, we get, on the event $\{\tau=0\}\subset\{F_0>0\}$,

$$
\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau=0\}}U_{\tau,T}\,\middle|\,\mathcal{G}_0\right) = \lim_{s\downarrow 0}\mathbb{1}_{\{\tau=0\}}\widehat{U}_{0,s,T} = \mathbb{1}_{\{\tau=0\}}\widehat{U}_{0,0,T}
$$

where (see (5.10))

$$
\widehat{U}_{0,0,T} = (F_0)^{-1}\,\mathbb{E}_{\mathbb{P}}(F_{0,T}U_{0,T}\,|\,\mathcal{F}_0) = \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{P}(\tau=0\,|\,\mathcal{F}_T)U_{0,T}\,|\,\mathcal{F}_0)}{\mathbb{P}(\tau=0\,|\,\mathcal{F}_0)}
$$

where $F_{0,T}=\mathbb{P}(\tau=0\,|\,\mathcal{F}_T)$. We conclude that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}(U_{\tau,T}\,|\,\mathcal{G}_0) &= \mathbb{1}_{\{\tau>0\}}\frac{1}{\mathbb{P}(\tau>0\,|\,\mathcal{F}_0)}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(0,\infty]}U_{u,T}\,d\mathbb{P}(\tau\le u\,|\,\mathcal{F}_T)\,\bigg|\,\mathcal{F}_0\bigg)\\
&\quad+\mathbb{1}_{\{\tau=0\}}\frac{1}{\mathbb{P}(\tau=0\,|\,\mathcal{F}_0)}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{[0]}U_{u,T}\,d\mathbb{P}(\tau\le u\,|\,\mathcal{F}_T)\,\bigg|\,\mathcal{F}_0\bigg).
\end{aligned}
$$

Remark 5.3.4. For $t=T$, we have that

$$
\mathbb{E}_{\mathbb{P}}(U_{\tau,T}\,|\,\mathcal{G}_T) = \mathbb{1}_{\{\tau>T\}}\widetilde{U}_{T,T} + \mathbb{1}_{\{\tau\le T\}}U_{\tau,T}, \tag{5.11}
$$

where formula (5.9) in Lemma 5.3.2 yields

$$
\widetilde{U}_{T,T} = \frac{\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>T\}}U_{\tau,T}\,\big|\,\mathcal{F}_T\big)}{\mathbb{P}(\tau>T\,|\,\mathcal{F}_T)} = (G_T)^{-1}\int_{(T,\infty]}U_{u,T}\,dF_{u,T}. \tag{5.12}
$$

Remark 5.3.5. Assume that $F_{u,t}$ satisfies the hypothesis (H). Then formula (5.10) simplifies as follows

$$\widehat{U}_{u,t,T} = (1-G_t)^{-1}\,\mathbb{E}_{\mathbb{P}}(F_{t,T}U_{u,T}\,|\,\mathcal{F}_t) = (F_t)^{-1}\,\mathbb{E}_{\mathbb{P}}(F_{t,t}U_{u,T}\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(U_{u,T}\,|\,\mathcal{F}_t).$$

In particular, if $\widehat{U}_{\tau,T} = g(\tau)$, we have that $\widehat{U}_{u,t,T} = g(u)$ and

$$\widetilde{U}_{t,T} = (G_t)^{-1}\,\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>t\}}g(\tau)\,|\,\mathcal{F}_t\big) = (G_t)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(t,\infty]} g(u)\,dF_u\,\bigg|\,\mathcal{F}_t\bigg).$$

5.3.2 Conditional Expectations for Pseudo-Initial Times

Under the extended density hypothesis, we establish the following counterpart of Lemma 5.3.2.

Lemma 5.3.3. *Let $U_{\cdot,T} : \mathbb{R}_+\times\Omega\to\mathbb{R}$ be a $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_T$-measurable map. If τ is a pseudo-initial time and the random variable $U_{\tau,T}$ is $\mathbb{P}$-integrable then, for every $t\in[0,T)$,*

$$\mathbb{E}_{\mathbb{P}}(U_{\tau,T}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\widetilde{U}_{t,T} + \mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,t,T}$$

where

$$\widetilde{U}_{t,T} = (G_t)^{-1}\,\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>t\}}U_{\tau,T}\,|\,\mathcal{F}_t\big) = (G_t)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(t,\infty]} U_{v,T}\,dF_{v,T}\,\bigg|\,\mathcal{F}_t\bigg)$$

and, for every $0\le u\le t<T$,

$$\widehat{U}_{u,t,T} = (m_{u,t})^{-1}\,\mathbb{E}_{\mathbb{P}}(m_{u,T}U_{u,T}\,|\,\mathcal{F}_t). \tag{5.13}$$

Proof. It suffices to revise the proof of Lemma 5.3.2 on the event $\{\tau\le t\}$. Let us first observe that, by Definition 5.2.5, for every $u\in\mathbb{R}_+$, the process $(m_{u,t})_{t\ge u}$ is a positive $(\mathbb{P},\mathbb{F})$-martingale and thus $\{m_{u,t}=0\}\subset\{m_{u,s}=0\}\subset\{m_{u,T}=0\}$ for all $u\le t\le s\le T$. Recall also that, by convention, we set $0/0=0$ and thus $\widehat{U}_{u,s,T}$ is well defined. Therefore, for all $t\le s\le T$ and any bounded, $\sigma(\tau)\vee\mathcal{F}_s$-measurable random variable $H_{\tau,s}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\le t\}}H_{\tau,s}U_{\tau,T}\big) &= \mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\le t\}}H_{\tau,s}U_{\tau,T}\,|\,\mathcal{F}_T\big)\Big) = \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,\infty]}\mathbb{1}_{\{u\le t\}}H_{u,s}U_{u,T}\,dF_{u,T}\bigg)\\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]} H_{u,s}m_{u,T}U_{u,T}\,dD_u\bigg) = \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]} H_{u,s}(m_{u,s})^{-1}\mathbb{E}_{\mathbb{P}}(m_{u,T}U_{u,T}\,|\,\mathcal{F}_s)m_{u,s}\,dD_u\bigg)\\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]} H_{u,s}\widehat{U}_{u,s,T}\,dF_{u,s}\bigg) = \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{\tau\le t\}}H_{\tau,s}\widehat{U}_{\tau,s,T}\Big).
\end{aligned}$$

By taking limit, and using similar arguments as in the proof of Lemma 5.3.2, we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\le t\}}U_{\tau,T}\,\middle|\,\mathcal{G}_t\right) &= \lim_{s\downarrow t}\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\tau\le t\}}U_{\tau,T}\,\middle|\,\sigma(\tau)\vee\mathcal{F}_s\right) = \lim_{s\downarrow t}\mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,s,T}\\
&= \lim_{s\downarrow t}\mathbb{1}_{\{\tau\le t\}}(m_{u,s})^{-1}\,\mathbb{E}_{\mathbb{P}}(m_{u,T}U_{u,T}\,|\,\mathcal{F}_s)_{u=\tau}\\
&= \mathbb{1}_{\{\tau\le t\}}(m_{u,t})^{-1}\,\mathbb{E}_{\mathbb{P}}(m_{u,T}U_{u,T}\,|\,\mathcal{F}_t)_{u=\tau} = \mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,t,T},
\end{aligned}$$

which proves (5.13). □

5.4 Properties of $\mathbb{G}$-Local Martingales

We consider the map $\widehat{U}:\mathbb{R}_+^2\times\Omega\to\mathbb{R}$ and we use the notation $(u,t,\omega)\mapsto\widehat{U}_{u,t}(\omega)$. We say that $\widehat{U}$ is an $\mathbb{F}$-*optional map* when it is $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{O}(\mathbb{F})$-measurable, where $\mathcal{O}(\mathbb{F})$ is the $\mathbb{F}$-optional σ-field in $\mathbb{R}_+\times\Omega$. In that case, the map $\widehat{U}_{\cdot,t}$ is $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_t$-measurable and the process $(\widehat{U}_{t,t})_{t\ge0}$ is $\mathbb{F}$-optional, in the usual sense. We will sometimes need an additional assumption that the process $(\widehat{U}_{t,t})_{t\ge0}$ is $\mathbb{F}$-predictable.

5.4.1 $\mathbb{G}$-Local Martingales for Pseudo-Honest Times

Consider an arbitrary random time τ such that $G = 1 - F$ is the Azéma supermartingale of τ. We denote by $G = M - A$ the Doob-Meyer decomposition of G. Then the dual $(\mathbb{P}, \mathbb{F})$-predictable projection of the indicator process $H_t = \mathbb{1}_{\{\tau \le t\}}$ satisfies $H^p = A$.

The following result, which corresponds to Propositions 5.1 and 5.6 in El Karoui et al. [36], is an important step towards establishing a $(\mathbb{P}, \mathbb{G})$-semimartingale decomposition of a $(\mathbb{P}, \mathbb{F})$-local martingale.

Theorem 5.4.1. *Assume that τ is a pseudo-honest time and $0 < F_{u,t} \le 1$ for every $0 < u \le t$. Let $\bar{U}$ be a $\mathbb{G}$-adapted and $\mathbb{P}$-integrable process given by the following expression*

$$\bar{U}_t = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t + \mathbb{1}_{\{\tau \le t\}} \widehat{U}_{\tau,t} \tag{5.14}$$

where $\widetilde{U}$ is an $\mathbb{F}$-adapted, $\mathbb{P}$-integrable process and $\widehat{U}$ is an $\mathbb{F}$-optional map such that for every $t \in \mathbb{R}_+$ the random variable $\widehat{U}_{\tau,t}$ is $\mathbb{P}$-integrable and the process $(\widehat{U}_t := \widehat{U}_{t,t})_{t \ge 0}$ is $\mathbb{F}$-predictable. Assume, in addition, that the following conditions are satisfied:
(i) the process $(W_t)_{t \ge 0}$ is a $(\mathbb{P}, \mathbb{F})$-local martingale where

$$W_t = \widetilde{U}_t G_t + \int_{(0,t]} \widehat{U}_v \, dF_v, \tag{5.15}$$

(ii) for any fixed $u, s \ge 0$, the process $(F_{s,t} \widehat{U}^0_{u,t})_{t \ge u \vee s}$ is a $(\mathbb{P}, \mathbb{F})$-local martingale where we denote $\widehat{U}^0_{u,t} = \widehat{U}_{u,t} - \widehat{U}_{u,u}$ for every $0 \le u \le t$.
Then the process $(\bar{U}_t)_{t \ge 0}$ is a $(\mathbb{P}, \mathbb{G})$-local martingale.

Proof. Since the proof proceeds along the similar lines as the proofs of Propositions 5.1 and 5.6 in El Karoui et al. [36], we will focus on computations and for the details regarding suitable localization and measurability arguments we refer to [36]. We start by noting that the following decomposition is valid

$$\bar{U}_t = \bar{U}_t \mathbb{1}_{\{\tau > t\}} + \bar{U}_\tau \mathbb{1}_{\{\tau \le t\}} + (\bar{U}_t - \bar{U}_\tau) \mathbb{1}_{\{\tau \le t\}} = \widetilde{U}_t \mathbb{1}_{\{\tau > t\}} + \widehat{U}_{\tau,\tau} \mathbb{1}_{\{\tau \le t\}} + (\widehat{U}_{\tau,t} - \widehat{U}_{\tau,\tau}) \mathbb{1}_{\{\tau \le t\}}.$$

It is thus enough to examine the following two subcases, corresponding to conditions (i) and (ii), respectively:
(a) the case of a process $\bar{U}$ stopped at τ,
(b) the case of a process $\bar{U}$ such that $\bar{U}_{\tau \wedge t} = 0$ for all $t \ge 0$.

Case (a). We first assume that a $\mathbb{G}$-adapted process $\bar{U}$ is stopped at τ, specifically,

$$\bar{U}_t = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t + \mathbb{1}_{\{\tau \le t\}} \widehat{U}_\tau \tag{5.16}$$

where $(\widetilde{U}_t)_{t \ge 0}$ is an $\mathbb{F}$-adapted process and $(\widehat{U}_t := \widehat{U}_{t,t})_{t \ge 0}$ is an $\mathbb{F}$-predictable process. We start by observing that, for every $0 \le s < t$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le t\}} \widehat{U}_\tau \,|\, \mathcal{G}_s) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{s < \tau \le t\}} \widehat{U}_\tau \,|\, \mathcal{G}_s) + \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le s\}} \widehat{U}_\tau \,|\, \mathcal{G}_s) \\
&= \mathbb{1}_{\{\tau > s\}} (G_s)^{-1} \, \mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}_v \, dH_v \,\bigg|\, \mathcal{F}_s \bigg) + \mathbb{1}_{\{\tau \le s\}} \widehat{U}_\tau \\
&= \mathbb{1}_{\{\tau > s\}} (G_s)^{-1} \, \mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}_v \, dA_v \,\bigg|\, \mathcal{F}_s \bigg) + \mathbb{1}_{\{\tau \le s\}} \widehat{U}_\tau \\
&= \mathbb{1}_{\{\tau > s\}} (G_s)^{-1} \, \mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}_v \, dF_v \,\bigg|\, \mathcal{F}_s \bigg) + \mathbb{1}_{\{\tau \le s\}} \widehat{U}_\tau .
\end{aligned}$$

Hence, for every $0 \le s < t$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(\bar{U}_t \,|\, \mathcal{G}_s) &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(\widetilde{U}_t G_t \,|\, \mathcal{F}_s) + \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}_v\, dF_v \,\bigg|\, \mathcal{F}_s\bigg) + \mathbb{1}_{\{\tau\le s\}}\widehat{U}_\tau \\ &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(W_t - W_s \,|\, \mathcal{F}_s) + \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(\widetilde{U}_s G_s \,|\, \mathcal{F}_s) + \mathbb{1}_{\{\tau\le s\}}\widehat{U}_\tau \\ &= \mathbb{1}_{\{\tau>s\}}\widetilde{U}_s + \mathbb{1}_{\{\tau\le s\}}\widehat{U}_\tau = \bar{U}_s\end{aligned}$$

where we used the assumption that the process W given by (5.15) is a $(\mathbb{P},\mathbb{F})$-martingale. We conclude that the process $\bar{U}$ given by (5.16) is a $(\mathbb{P},\mathbb{G})$-martingale.

Case (b). Let us denote $\widehat{U}^0_{v,t} = \widehat{U}_{v,t} - \widehat{U}_{v,v}$ for $0 \le v \le t$. Consider a $\mathbb{G}$-adapted process $\bar{U}$ given by $\bar{U}_t = \mathbb{1}_{\{\tau\le t\}}\widehat{U}^0_{\tau,t}$ where $\widehat{U}^0_{t,t} = 0$. We need to show that the equality $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}\widehat{U}^0_{\tau,t} \,|\, \mathcal{G}_s) = \mathbb{1}_{\{\tau\le s\}}\widehat{U}^0_{\tau,s}$ holds for every $0 \le s < t$. From part (i) in Lemma 5.3.2, we obtain, for every $0 \le s < t$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}\widehat{U}^0_{\tau,t} \,|\, \mathcal{G}_s) &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}^0_{v,t}\, dF_{v,t} \,\bigg|\, \mathcal{F}_s\bigg) \\ &\quad + \mathbb{1}_{\{\tau\le s\}}(F_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(F_{s,t}\widehat{U}^0_{u,t} \,|\, \mathcal{F}_s)_{|u=\tau} = I_1 + I_2.\end{aligned}$$

Let us examine I_1. We first assume that $s > 0$. Recall that we assume that the hypothesis (HP) holds and $0 < F_{v,t} \le 1$ for every $0 < v \le t$. Hence, for $0 < s \le v \le t$, we can write $dF_{v,t} = F_{s,t}\, d(F_{v,v}F^{-1}_{s,v}) = F_{s,t}\, dD^s_v$ where the process $(D^s_v = F_{v,v}F^{-1}_{s,v})_{v\ge s}$ is increasing and $\mathbb{F}$-adapted. Consequently,

$$\begin{aligned}I_1 &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \mathbb{E}_{\mathbb{P}}(F_{s,t}\widehat{U}^0_{v,t} \,|\, \mathcal{F}_v)\, dD^s_v \,\bigg|\, \mathcal{F}_s\bigg) \\ &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} F_{s,v}\widehat{U}^0_{v,v}\, dD^s_v \,\bigg|\, \mathcal{F}_s\bigg) = 0\end{aligned}$$

where we first used condition (ii) and subsequently the equality $\widehat{U}^0_{v,v} = 0$. It remains to examine the case $s = 0$. We denote $\widehat{U}^{0+}_{\tau,t} = \max(\widehat{U}^0_{\tau,t}, 0)$ and $\widehat{U}^{0-}_{\tau,t} = \max(-\widehat{U}^0_{\tau,t}, 0)$. Then, for all $t > 0$,

$$\begin{aligned}I_1 &= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(0,t]} \widehat{U}^0_{v,t}\, dF_{v,t} \,\bigg|\, \mathcal{F}_0\bigg) = \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{s\le\tau\le t\}}\widehat{U}^0_{\tau,t} \,|\, F_t\Big) \,\bigg|\, \mathcal{F}_0\bigg) \\ &= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{s\le\tau\le t\}}\widehat{U}^{0+}_{\tau,t} \,|\, F_t\Big) \,\bigg|\, \mathcal{F}_0\bigg) - \mathbb{E}_{\mathbb{P}}\bigg(\lim_{\epsilon\downarrow 0}\mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{s\le\tau\le t\}}\widehat{U}^{0-}_{\tau,t} \,|\, F_t\Big) \,\bigg|\, \mathcal{F}_0\bigg).\end{aligned}$$

By the monotone convergence theorem for conditional expectations, we obtain

$$\begin{aligned}I_1 &= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\,\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}^0_{v,t}\, dF_{v,t} \,\bigg|\, \mathcal{F}_0\bigg) \\ &= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\,\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \mathbb{E}_{\mathbb{P}}(F_{s,t}\widehat{U}^0_{v,t} \,|\, \mathcal{F}_u)\, dD^s_v \,\bigg|\, \mathcal{F}_0\bigg) = 0\end{aligned}$$

where we used condition (ii) and the equality $\widehat{U}^0_{v,v} = 0$. For I_2, using again condition (ii), we obtain, for $0 \le u \le s < t$,

$$I_2 = \mathbb{1}_{\{\tau\le s\}}(F_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(F_{s,t}\widehat{U}^0_{u,t} \,|\, \mathcal{F}_s)_{|u=\tau} = \mathbb{1}_{\{\tau\le s\}}(F_s)^{-1}\,F_{s,s}(\widehat{U}^0_{u,s})_{|u=\tau} = \mathbb{1}_{\{\tau\le s\}}\widehat{U}^0_{\tau,s}.$$

We conclude that the process $\big(\mathbb{1}_{\{\tau\le t\}}\widehat{U}^0_{\tau,t}\big)_{t\ge 0}$ is a $(\mathbb{P},\mathbb{G})$-martingale and thus the proof of the proposition is completed. ☐

The following corollary to Theorem 5.4.1 deals with the special case when the process U given by (5.14) is continuous at τ. It is easy to check that under the assumptions of Corollary 5.4.1 the process $(\widehat{U}_t := \widehat{U}_{t,t})_{t\ge 0}$ is $\mathbb{F}$-predictable.

Corollary 5.4.1. *Under the assumptions of Theorem 5.4.1 we postulate, in addition, that the equality* $\widetilde{U}_{t-} = \widehat{U}_{t,t}$ *holds for every* $t \in \mathbb{R}_+$. *Then the process* $\bar{U}$ *is continuous at* τ *and condition* (*i*) *in Theorem 5.4.1 can be replaced by the following condition:*
(*i*′) *the process* $(W_t)_{t\geq 0}$ *is a* $(\mathbb{P},\mathbb{F})$*-local martingale where*

$$W_t = \widetilde{U}_t G_t + \int_{(0,t]} \widetilde{U}_{u-}\, dF_u. \tag{5.17}$$

To establish another corollary to Theorem 5.4.1, we assume that $F_{u,t}$ is completely separable.

Corollary 5.4.2. *Under the assumptions of Theorem 5.4.1 we postulate, in addition, that the* $(\mathbb{P},\mathbb{F})$*-conditional distribution of* τ *satisfies* $F_{u,t} = K_u L_t$ *for all* $0 \leq u \leq t$, *where* K *is a positive,* $\mathbb{F}$*-adapted, increasing process and* L *is a positive* $(\mathbb{P},\mathbb{F})$*-martingale. Then condition (ii) in Theorem 5.4.1 can be replaced by the following condition:*
(ii') For every $u \geq 0$, *the process* $(W_{u,t} = L_t \widehat{U}^0_{u,t})_{t\geq u}$ *is a* $(\mathbb{P},\mathbb{F})$*-martingale.*

Proof. We observe that the computations for case (b) in the proof of Proposition 5.4.1 can be simplified. Using part (ii) in Lemma 5.3.2, we obtain, for every $0 \leq s < t$ (note that $L_t = 0$ on the event $\{L_s = 0\}$)

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq t\}}\widehat{U}^0_{\tau,t}\,|\,\mathcal{G}_s) &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} L_t\widehat{U}^0_{u,t}\,dK_u\,\bigg|\,\mathcal{F}_s\bigg) + \mathbb{1}_{\{\tau\leq s\}}(L_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(L_t\widehat{U}^0_{u,t}\,|\,\mathcal{F}_s)_{|u=\tau}\\
&= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \mathbb{E}_{\mathbb{P}}(W_{u,t}\,|\,\mathcal{F}_u)\,dK_u\,\bigg|\,\mathcal{F}_s\bigg) + \mathbb{1}_{\{\tau\leq s\}}(L_s)^{-1}\,\mathbb{E}_{\mathbb{P}}(W_{u,t}\,|\,\mathcal{F}_s)_{|u=\tau}\\
&= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} W_{u,u}\,dK_u\,\bigg|\,\mathcal{F}_s\bigg) + \mathbb{1}_{\{\tau\leq s\}}(L_s)^{-1}(W_{u,s})_{|u=\tau} = \mathbb{1}_{\{\tau\leq s\}}\widehat{U}^0_{\tau,s}
\end{aligned}$$

where we used condition (ii') in the penultimate equality and the equality $W_{u,u} = L_u\widehat{U}^0_{u,u} = 0$ in the last one. □

5.4.2 $\mathbb{G}$-Local Martingales for Pseudo-Initial Times

It was necessary to assume in Theorem 5.4.1 that the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is non-degenerate, since the random measure $D^s_u := F_{u,u}(F_{s,u})^{-1}$ is not always well defined when the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is degenerate. In order to circumvent this technical assumption, one can postulate instead that $F_{u,t}$ satisfies the hypothesis (*ED*). In the next result, we work under the setup of Theorem 5.4.1, but we no longer assume that $0 < F_{u,t} \leq 1$ for every $0 < u \leq t$.

Proposition 5.4.1. *Suppose that* τ *is a pseudo-initial time and* (5.5) *holds with a positive random field* $(m_{s,t})_{t\geq s}$ *and an* $\mathbb{F}$*-adapted increasing process* D. *Then condition (ii) in Theorem 5.4.1 can be replaced by the following condition:*
(ii) for every* $u \geq 0$, *the process* $(m_{u,t}\widehat{U}^0_{u,t})_{t\geq u}$ *is a* $(\mathbb{P},\mathbb{F})$*-local martingale.*

Proof. We only need to adjust the proof of Theorem 5.4.1 in case (b). Let $\bar{U}_t = \mathbb{1}_{\{\tau\leq t\}}\widehat{U}^0_{\tau,t}$ where $\widehat{U}^0_{t,t} = 0$. Using Lemma 5.3.3, we obtain, for every $0 \leq s < t$ (recall that $\{m_{u,s} = 0\} \subset \{m_{u,t} = 0\}$)

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq t\}}\widehat{U}^0_{\tau,t}\,|\,\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}^0_{u,t}\,dF_{u,t}\,\bigg|\,\mathcal{F}_s\bigg)&\\
+ \mathbb{1}_{\{\tau\leq s\}}\,(m_{u,s})^{-1}\mathbb{E}_{\mathbb{P}}(m_{u,t}\widehat{U}^0_{u,t}\,|\,\mathcal{F}_s)_{|u=\tau} = I_1 + I_2.&
\end{aligned}$$

The integral I_1 satisfies

$$I_1 = \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \widehat{U}^0_{u,t}\,dF_{u,t}\,\bigg|\,\mathcal{F}_s\bigg) = \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} m_{u,t}\widehat{U}^0_{u,t}\,dD_u\,\bigg|\,\mathcal{F}_s\bigg)$$
$$= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\,\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]} \mathbb{E}_{\mathbb{P}}\big(m_{u,t}\widehat{U}^0_{u,t}\,|\,\mathcal{F}_u\big)\,dD_u\,\bigg|\,\mathcal{F}_s\bigg) = 0$$

where to obtain the last equality we first used assumption (ii*) and next the equality $\widehat{U}^0_{u,u} = 0$. The integral I_2 simplifies to

$$I_2 = \mathbb{1}_{\{\tau\le s\}}(m_{u,s})^{-1}\,\mathbb{E}_{\mathbb{P}}(m_{u,t}\widehat{U}^0_{u,t}\,|\,\mathcal{F}_s)\big|_{u=\tau} = \mathbb{1}_{\{\tau\le s\}}(m_{u,s})^{-1}\,m_{u,s}(\widehat{U}^0_{u,s})\big|_{u=\tau} = \mathbb{1}_{\{\tau\le s\}}\widehat{U}_{\tau,s}.$$

We conclude that the process $\bar{U}$ is a $(\mathbb{P},\mathbb{G})$-martingale, as was required to show. □

5.5 Compensators of the Indicator Process

Our next goal is to compute the $(\mathbb{P},\mathbb{G})$-dual predictable projection of the indicator process $H_t = \mathbb{1}_{\{\tau\le t\}}$ where, as usual, we denote by $\mathbb{G}$ the progressive enlargement of $\mathbb{F}$ with a random time τ. Recall that the Doob-Meyer decomposition of G is denoted by $G = M - A$. Then the $(\mathbb{P},\mathbb{F})$-dual predictable projection (i.e., the $(\mathbb{P},\mathbb{F})$-*compensator*) of H, denoted as H^p, coincides with the $\mathbb{F}$-predictable, increasing process A. To find the $(\mathbb{P},\mathbb{G})$-dual predictable projection (i.e., the $(\mathbb{P},\mathbb{G})$-*compensator*) of H, it is enough to apply the following classic result, due to Jeulin and Yor [65] (see also Guo and Zeng [47]), and to compute explicitly the $(\mathbb{P},\mathbb{F})$-compensator of H.

Theorem 5.5.1. *Let τ be a random time with the Azéma supermartingale G. Then the $(\mathbb{P},\mathbb{G})$-compensator of H equals*

$$H^{p,\mathbb{G}}_t = \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\,dH^p_u, \tag{5.18}$$

meaning that the process $H - H^{p,\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$-martingale.

5.5.1 Compensator of H under Complete Separability

Let us first examine the case where the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ under $\mathbb{P}$ is completely separable, that is, $F_{u,t} = K_uL_t$. We assume, in addition, that the increasing process K is $\mathbb{F}$-predictable. It is worth noting that both assumptions are satisfied in the construction of τ based on a predictable multiplicative system, provided that $G_t < 1$ for all $t > 0$ (see [74]). By applying the integration by parts formula to F and using the assumption that K is an $\mathbb{F}$-predictable process, we obtain

$$F_t = K_tL_t = K_0L_0 + \int_{(0,t]} K_u\,dL_u + \int_{(0,t]} L_{u-}\,dK_u.$$

Hence, by the uniqueness of the Doob-Meyer decomposition, we conclude that $dA_u = L_{u-}dK_u$. Consequently, using the Jeulin-Yor formula (5.18), we obtain

$$H^{p,\mathbb{G}}_t = \int_{(0,t\wedge\tau]} \frac{L_{u-}}{1 - L_{u-}K_{u-}}\,dK_u.$$

5.5.2 Compensator of H for a Pseudo-Initial Time

Assume now that τ is a pseudo-initial time, that is, the hypothesis (ED) holds. Let the process m be given as $(m_t := m_{t,t})_{t\ge0}$ and let ${}^pm$ denote the $(\mathbb{P},\mathbb{F})$-predictable projection of m.

Proposition 5.5.1. *Assume that τ is a pseudo-initial time and the increasing process D in formula (5.5) is $\mathbb{F}$-predictable. Then the $(\mathbb{P},\mathbb{F})$-compensator of H is given by*

$$H_t^p = \int_{(0,t]} {}^p m_u \, dD_u. \tag{5.19}$$

Proof. It suffices to compute the Doob-Meyer decomposition of F. From the hypothesis (*ED*), it is easy to see that

$$F_t = m_{0,t}\Delta D_0 + \int_{(0,t]} (m_{u,t} - m_{u,u}) \, dD_u + \int_{(0,t]} m_u \, dD_u. \tag{5.20}$$

The process D is $\mathbb{F}$-predictable and increasing, and thus

$$\left(\int_{(0,t]} m_u \, dD_u \right)^p = \int_{(0,t]} {}^p m_u \, dD_u.$$

We claim that the Doob-Meyer decomposition of F is given by

$$\begin{aligned} F_t &= m_{0,t}\Delta D_0 + \int_{(0,t]} (m_{u,t} - m_{u,u}) \, dD_u + \int_{(0,t]} (m_u - {}^p m_u) \, dD_u + \int_{(0,t]} {}^p m_u \, dD_u \\ &= M_t^1 + M_t^2 + M_t^3 + A_t \end{aligned}$$

where A is an $\mathbb{F}$-predictable increasing process given by (5.19) and the processes M^i for $i = 1, 2, 3$ are given by

$$M_t^1 = m_{0,t}\Delta D_0, \quad M_t^2 = \int_{(0,t]} (m_{u,t} - m_{u,u}) \, dD_u, \quad M_t^3 = \int_{(0,t]} (m_u - {}^p m_u) \, dD_u.$$

Since the processes M^1 and M^3 are clearly $(\mathbb{P},\mathbb{F})$-martingales, it remains to show that M^2 is a $(\mathbb{P},\mathbb{F})$-martingale as well. We have, for all $0 \le s \le t$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left(M_t^2 \,\middle|\, \mathcal{F}_s \right) &= \mathbb{E}_{\mathbb{P}} \left(\int_{(0,t]} (m_{u,t} - m_{u,u}) \, dD_u \,\middle|\, \mathcal{F}_s \right) \\ &= \mathbb{E}_{\mathbb{P}} \left(\int_{(0,s]} (m_{u,t} - m_{u,u}) \, dD_u \,\middle|\, \mathcal{F}_s \right) + \mathbb{E}_{\mathbb{P}} \left(\int_{(s,t]} (m_{u,t} - m_{u,u}) \, dD_u \,\middle|\, \mathcal{F}_s \right) \\ &= \int_{(0,s]} \mathbb{E}_{\mathbb{P}} \left(m_{u,t} - m_{u,u} \,\middle|\, \mathcal{F}_s \right) \, dD_u + \mathbb{E}_{\mathbb{P}} \left(\int_{(s,t]} \mathbb{E}_{\mathbb{P}} \left(m_{u,t} - m_{u,u} \,\middle|\, \mathcal{F}_u \right) \, dD_u \,\middle|\, \mathcal{F}_s \right) \\ &= \int_{(0,s]} (m_{u,s} - m_{u,u}) \, dD_u = M_s^2, \end{aligned}$$

and thus the process M^2 is a $(\mathbb{P},\mathbb{F})$-martingale. □

In the next result, we no longer assume the process D is predictable. We impose instead some additional conditions on the process m. Without loss of generality, we may and do assume that D is an $\mathbb{F}$-optional process. This is because D can always be regularized to be càdlàg and $\mathbb{F}$-adapted, since the filtration $\mathbb{F}$ is right-continuous and D is non-decreasing.

Proposition 5.5.2. *Assume that τ is a pseudo-initial time and the process m is a special semi-martingale with the canonical decomposition $m = N + P$. If the predictable covariation of N and D exists then the $(\mathbb{P},\mathbb{F})$-compensator of H is given by the formula*

$$H_t^p = \langle N, D \rangle_t + \int_{(0,t]} {}^p m_u \, dD_u^p \tag{5.21}$$

where ${}^p m_u = N_{u-} + P_u$.

Proof. Once again, we compute the Doob-Meyer decomposition of F starting from (5.20). Using the canonical decomposition of m, we obtain

$$\begin{aligned} F_t &= m_{0,t}\Delta D_0 + \int_{(0,t]} (m_{u,t} - m_{u,u})\, dD_u - \int_{(0,t]} (N_t - N_u)\, dD_u + N_t D_t + \int_{(0,t]} P_u\, dD_u \\ &= M_t^1 + M_t^2 - \widehat{M}_t^3 + [N,D]_t + \int_{(0,t]} (N_{u-} + P_u)\, dD_u \end{aligned}$$

where the second equality is obtained by an application of the integration by parts formula to the product ND and upon setting

$$M_t^1 = m_{0,t}\Delta D_0, \quad M_t^2 = \int_{(0,t]} (m_{u,t} - m_{u,u})\, dD_u,$$

and

$$\widehat{M}_t^3 = \int_{(0,t]} (N_t - N_u)\, dD_u + \int_{(0,t]} D_{u-}\, dN_u.$$

From the proof of Proposition 5.5.1, we know that the processes M^1 and M^2 are $(\mathbb{P},\mathbb{F})$-martingales. Similar arguments as used for M^2 show that the process $\widehat{M}^3$ is a $(\mathbb{P},\mathbb{F})$-martingale as well. Using the dual predictable projection of D, we can write

$$\int_{(0,t]} (N_{u-} + P_u)\, dD_u = \int_{(0,t]} (N_{u-} + P_u)\, d(D_u - D_u^p) + \int_{(0,t]} (N_{u-} + P_u)\, D_u^p.$$

Finally, the $\mathbb{F}$-predictable covariation of N and D is assumed to exist and, obviously,

$$[N,D] = [N,D] - \langle N,D\rangle + \langle N,D\rangle\,.$$

We conclude that the $\mathbb{F}$-predictable, increasing process A in the Doob-Meyer decomposition of F is given by

$$A_t = \langle N,D\rangle_t + \int_{(0,t]} (N_{u-} + P_u)\, dD_u^p.$$

To complete the proof of equality (5.21), it suffices to observe that ${}^p m_u = N_{u-} + P_u$. □

5.6 Hypothesis (H') and Semimartingale Decompositions

The aim of this section is to analyze the validity of the classic hypothesis (H') for progressive enlargements associated with pseudo-honest and pseudo-initial times. We establish here the main results of this work, Theorems 5.6.2 and 5.6.6, and we study the case of the multiplicative construction of a random time associated with a predetermined Azéma submartingale.

Let us first recall the general definition, in which $\mathbb{K}$ stands for any enlargement of $\mathbb{F}$, that is, any filtration such that $\mathbb{F} \subset \mathbb{K}$.

Definition 5.6.1. The *hypothesis* (H') is said to hold for $\mathbb{F}$ and its enlargement $\mathbb{K}$ whenever any $(\mathbb{P},\mathbb{F})$-semimartingale is also a $(\mathbb{P},\mathbb{K})$-semimartingale.

For exhaustive studies of the hypothesis (H') the interested reader is referred to Jeulin [63], who examined a general case as well as honest times, and Jacod [50], who worked under the density hypothesis and covered the initial times. The latter study was recently extended by Kchia and Protter [71], who dealt with the progressive enlargement with a general stochastic process, and not only the indicator process of a random time.

As is well known, to establish the hypothesis (H') between $\mathbb{F}$ and any enlargement $\mathbb{K}$, it suffices to show that any bounded $(\mathbb{P},\mathbb{F})$-martingale is a $(\mathbb{P},\mathbb{K})$-semimartingale (see Yor [107]). This crucial observation follows, for instance, from the Jacod-Mémin decomposition of a $(\mathbb{P},\mathbb{F})$-semimartingale; $X = X_0 + K + B + N$ where K represents large jumps, B is predictable of finite variation and N is a local martingale with jumps bounded by 1 (see, e.g., page 3 in Jeulin [63]). One can then show that, under the hypothesis (H'), any bounded $(\mathbb{P},\mathbb{F})$-martingale is in fact a special $(\mathbb{P},\mathbb{K})$-semimartingale and this in turn implies that, more generally, any special $(\mathbb{P},\mathbb{F})$-semimartingale remains a special $(\mathbb{P},\mathbb{K})$-semimartingale. Therefore, assuming that the hypothesis (H') holds for $\mathbb{F}$ and $\mathbb{K}$, the natural goal is thus to find the canonical semimartingale decomposition with respect to the enlarged filtration $\mathbb{K}$ of a bounded $(\mathbb{P},\mathbb{F})$-martingale. In addition, if the hypothesis (H') for $\mathbb{F}$ and $\mathbb{K}$ fails to hold then the goal is to describe the class of $(\mathbb{P},\mathbb{F})$-semimartingales that remain also $(\mathbb{P},\mathbb{K})$-semimartingales. For an exhaustive study of these problems, the reader may consult Jeulin [63].

Let us now focus on the case of the progressive enlargement of a filtration $\mathbb{F}$ with a random time. It is well known, in particular, that for any random time τ with values in $\bar{\mathbb{R}}_+$ and any $(\mathbb{P},\mathbb{F})$-local martingale U, the stopped process U^τ is a $(\mathbb{P},\mathbb{G})$-semimartingale (see Yor [107] and Jeulin and Yor [65]). For the future reference, we first state a theorem, which recalls and combines results from papers by Jeulin and Yor [65] (for part (i), see Theorem 1 in [65]) and Jeulin [63] (for part (ii), see Proposition 4.16 in [63]).

Let us quote a result from Section 2.3

Theorem 5.6.1. *Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ with an arbitrary random time τ. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the stopped process U^τ is a $(\mathbb{P},\mathbb{G})$-special semimartingale.*
(i) The process

$$U_{t\wedge\tau} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d(\langle U, M\rangle_u + \breve{U}^p_u) \tag{5.22}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale, where $\breve{U}^p$ stands for the dual $\mathbb{F}$-predictable projection of the process $\breve{U}_t = \Delta U_\tau \mathbb{1}_{\{\tau\le t\}}$.
(ii) The process

$$U_{t\wedge\tau} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\langle U, \bar{M}\rangle_u \tag{5.23}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale, where $\bar{M}$ is the unique BMO martingale such that $\mathbb{E}_{\mathbb{P}}(N_\tau) = \mathbb{E}_{\mathbb{P}}(N_\infty \bar{M}_\infty)$ for every bounded $(\mathbb{P},\mathbb{F})$-martingale N.

Observe that (5.22) and (5.23) yield alternative representations for the canonical decomposition of the special $(\mathbb{P},\mathbb{G})$-semimartingale U^τ. Hence, from the uniqueness of the canonical decomposition of a special semimartingale, we deduce the equality

$$\int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d(\langle U, M\rangle_u + \breve{U}^p_u) = \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\left\langle U, \bar{M}\right\rangle_u \tag{5.24}$$

which necessarily holds for an arbitrary $(\mathbb{P},\mathbb{F})$-local martingale U.

Remark 5.6.1. It is known that $G = \bar{M} - \bar{A}$ where $\bar{A} = H^o$ is the dual $\mathbb{F}$-optional projection of H. Under the assumption (**C**) (that is, when all $\mathbb{F}$-martingales are continuous) and/or the assumption (**A**) (that is, when the random time τ avoids all $\mathbb{F}$-stopping times), we have that $\bar{M} = M$ and thus also $H^p = H^o$ or, equivalently, $A = \bar{A}$.

We will also need the following auxiliary result (see Lemma 4 in Jeulin and Yor [65]).

Lemma 5.6.1. *Let U be a $(\mathbb{P},\mathbb{F})$-local martingale. Denote by $\breve{U}^p$ the dual $(\mathbb{P},\mathbb{F})$-predictable projection of the process*

$$\breve{U}_t = \int_{(0,t]} \Delta U_s\, dH_s = \Delta U_\tau \mathbb{1}_{\{\tau\le t\}}.$$

Then the process

$$\int_{(0,t]} \Delta U_s\, dH_s - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\breve{U}^p_u = \Delta U_\tau \mathbb{1}_{\{\tau\le t\}} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\breve{U}^p_u \tag{5.25}$$

is a $(\mathbb{P},\mathbb{G})$*-local martingale.*

5.6.1 Hypothesis (H') for the Progressive Enlargement

Our goal is to show that if a random time τ given on some filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ satisfies the hypothesis (HP) and its $(\mathbb{P},\mathbb{F})$-conditional distribution is positive then the $\mathbb{G}$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-semimartingale can be computed explicitly. From Theorem 5.6.1, we know that for any $(\mathbb{P},\mathbb{F})$-local martingale U, the stopped process U^τ is a $(\mathbb{P},\mathbb{G})$-special semimartingale with explicitly known canonical decomposition. Therefore, it will be enough to focus on the behavior of a process U after τ. It should be stressed that we do not claim that the hypothesis (HP) implies that the hypothesis (H') between $\mathbb{F}$ and the progressive enlargement $\mathbb{G}$ holds, in general, since certain additional assumptions will be imposed when deriving alternative versions of $\mathbb{G}$-semimartingale decompositions of a $(\mathbb{P},\mathbb{F})$-local martingale. Theorem 5.6.2 furnishes an explicit $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-local martingale U when τ is a pseudo-honest time. This result can be seen as a counterpart of Proposition 5.9 in El Karoui et al. [36] who dealt with the case of an initial time (see also Jeanblanc and Le Cam [57] who examined both initial and honest times). In the special case of the multiplicative construction (see Corollary 5.6.3 and Remark 5.6.5), it is also related to Theorem 7.1 in Jeanblanc and Song [59].

Theorem 5.6.2. *Assume that* τ *is a pseudo-honest time such that, for every* $s\geq 0$*, the bounded* $(\mathbb{P},\mathbb{F})$*-martingale* $(F_{s,u})_{u\geq s}$ *is strictly positive. If* U *is a* $(\mathbb{P},\mathbb{F})$*-local martingale then the process* $\bar{U}$ *is a* $(\mathbb{P},\mathbb{G})$*-local martingale where*

$$\bar{U}_t = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d(\langle U,M\rangle_u + \breve{U}^p_u) - \int_{(t\wedge s,t]} (F_{s,u-})^{-1}\, d\langle U, F_{s,\cdot}\rangle_u \Big|_{s=\tau}. \tag{5.26}$$

Hence U *is a* $(\mathbb{P},\mathbb{G})$*-special semimartingale and equality* (5.26) *yields its canonical decomposition.*

Note that $(t\wedge\tau,t]=\emptyset$ on the event $\{\tau\geq t\}$, since manifestly $(t\wedge s,t]=\emptyset$ for all $s\geq t$. Before proceeding to the proof of Theorem 5.6.2, we make some pertinent remarks and prove a preliminary lemma.

Remark 5.6.2. Recall that any local martingale is locally in the space $\mathcal{H}^1$. If N is a BMO martingale then, by Fefferman's inequality (see Revuz and Yor [97]), there exists a constant c such that for any local martingale U

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^\infty |d[U,N]_t|\right) \leq c\, \|U\|_{\mathcal{H}^1}\|N\|_{BMO}.$$

Consequently, the process $[U,N]$ is locally of integrable variation and its compensator $\langle U,N\rangle$ is well defined.

Remark 5.6.3. The Azéma's supermartingale G is generated by the $\mathbb{F}$-predictable, increasing process A, in the sense that, for every $t\geq 0$,

$$G_t = \mathbb{E}_{\mathbb{P}}\left(A_\infty \,\middle|\, \mathcal{F}_t\right) - A_t = M_t - A_t.$$

This implies that the process $M_t = \mathbb{E}_{\mathbb{P}}\left(A_\infty \,\middle|\, \mathcal{F}_t\right)$ is a BMO martingale since $G\leq 1$ (see Proposition 10.13 in [48]). It is also known that any bounded martingale is a BMO martingale (see Proposition 10.11 in [48]).

As a first step towards the proof of Theorem 5.6.2, we establish the existence of the integrals appearing in right-hand side of (5.26).

Lemma 5.6.2. *Under the assumptions of Theorem 5.6.2, the integrals in the right-hand side of equality* (5.27) *are well defined.*

Proof. The process $(G_t)^{-1}\mathbb{1}_{\{\tau>t\}}$ is known to have, with probability 1, finite left-hand limits for all $t\in\mathbb{R}_+$ (see Yor [107]) and thus it possess a finite left-hand limit at τ. Hence the first integral in (5.26) is a well defined $\mathbb{G}$-adapted process of finite variation. Next, let us check that for all $u\le t$ the integral $Z_{u,t}=\int_{(u,t]}(F_{u,v-})^{-1}d\left\langle U,F_{u,\cdot}\right\rangle_v$ is well defined. We proceed as follows. Under the standing assumption that $(F_{u,t})_{t\ge u}$ is a strictly positive process, the stochastic logarithm

$$\mathcal{L}(F_{u,\cdot})=\int_{(u,t]}(F_{u,v-})^{-1}dF_{u,v}$$

is well defined. For the existence of the predictable bracket $\langle U,\mathcal{L}(F_{u,\cdot})\rangle$, it is sufficient to check that the càdlàg process $\mathcal{L}(F_{u,\cdot})$ is a locally bounded martingale and for this purpose it is enough to show that the jump process $\Delta\mathcal{L}(F_{u,\cdot})_t=(F_{u,t-})^{-1}\Delta F_{u,t}$ is locally bounded for $t\ge u$. The latter property is clear since the left-continuous process $(F_{u,t-})^{-1}$ is locally bounded (we use here the property that $(F_{u,t})_{t\ge u}$ is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale) and the jumps of $F_{u,\cdot}$ are obviously bounded by 1. We conclude that the integral $Z_{u,t}$ is well defined since

$$Z_{u,t}=\int_{(u,t]}(F_{u,v-})^{-1}d\langle U,F_{u,\cdot}\rangle_v=\int_{(u,t]}d\langle U,\mathcal{L}(F_{u,\cdot})\rangle_v.$$

Moreover, the process $(Z_{u,t})_{t\ge u}$ is of locally integrable variation, since

$$\mathbb{E}_{\mathbb{P}}\left(\int_u^\infty|d\langle U,\mathcal{L}(F_{u,\cdot})\rangle_v|\right)\le c\,\|U\|_{\mathcal{H}^1}\|\mathcal{L}(F_{u,\cdot})\|_{BMO},$$

where the local martingale U is locally in $\mathcal{H}^1$ and the locally bounded martingale $\mathcal{L}(F_{u,\cdot})$ is locally in the space BMO. □

We are now in a position to prove Theorem 5.6.2. Note that Theorem 5.6.1 is not employed in the proof of Theorem 5.6.2, although we use Lemma 5.6.2 to compensate the jump of U at τ.

Proof of Theorem 5.6.2. Although the present set-up is more general than the one studied in El Karoui et al. [36], some steps in our proof are analogous to those employed in the proof of Proposition 5.9 in [36]. We start by noting that we may and do assume, without loss of generality, that U is a uniformly integrable $(\mathbb{P},\mathbb{F})$-martingale. In view of Lemma 5.6.1, it suffices to show that if U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process $\bar U$ is a $(\mathbb{P},\mathbb{G})$-local martingale where

$$\bar U_t=U_t-\int_{(0,t\wedge\tau]}(G_{u-})^{-1}\,d\langle U,M\rangle_u-\Delta U_\tau\mathbb{1}_{\{\tau\le t\}}-\int_{(t\wedge s,t]}(F_{s,u-})^{-1}\,d\langle U,F_{s,\cdot}\rangle_u\Big|_{s=\tau}. \tag{5.27}$$

We note that $\bar U$ satisfies the equality $\bar U=U-B^*$ where the process B^* is defined by

$$B^*_t=\widetilde B_t\mathbb{1}_{\{\tau>t\}}+\widehat B_{\tau,t}\mathbb{1}_{\{\tau\le t\}}$$

where in turn the $\mathbb{F}$-predictable process $\widetilde B$ equals

$$\widetilde B_t=\int_{(0,t]}(G_{u-})^{-1}\,d\langle U,M\rangle_u \tag{5.28}$$

and the map $\widehat B$ is given by, for all $0\le u\le t$,

$$\widehat B_{u,t}=\widetilde B_u+\Delta U_u+\int_{(u,t]}(F_{u,v-})^{-1}\,d\langle U,F_{u,\cdot}\rangle_v$$

where $\Delta U_u=U_u-U_{u-}$. We need to show that the process

$$\bar U_t=\mathbb{1}_{\{\tau>t\}}\widetilde U_t+\mathbb{1}_{\{\tau\le t\}}\widehat U_{\tau,t}=(U_t-\widetilde B_t)\mathbb{1}_{\{\tau>t\}}+(U_t-\widehat B_{\tau,t})\mathbb{1}_{\{\tau\le t\}}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale. We observe that $\widehat{U}_{u,t} = U_t - \widehat{B}_{u,t}$ is an $\mathbb{F}$-optional map and the process $\widehat{U}_{t,t} = U_{t-} - \widetilde{B}_t$ is $\mathbb{F}$-predictable. Hence the assumptions of Theorem 5.4.1 are satisfied.

Therefore, in order to show that $\bar{U}$ is a $(\mathbb{P},\mathbb{G})$-local martingale, it suffices to show that: (i) the process

$$(U_t - \widetilde{B}_t)G_t - \int_{(0,t]} (U_{u-} - \widetilde{B}_u)\, dG_u \tag{5.29}$$

is a $(\mathbb{P},\mathbb{F})$-local martingale and (ii) for any fixed $u, s \geq 0$, the process $(F_{s,t}\widehat{U}^0_{u,t})_{t\geq s\vee u}$ is a $(\mathbb{P},\mathbb{F})$-local martingale. For brevity, we will use the notation $X \stackrel{mart}{=} Y$ whenever the process $X - Y$ is a $(\mathbb{P},\mathbb{F})$-local martingale.

To establish (i), we observe that

$$\begin{aligned} d\big((U_t - \widetilde{B}_t)G_t\big) &= U_{t-}\, dG_t + G_{t-}\, dU_t + d[U,G]_t - \widetilde{B}_t\, dG_t - G_{t-}\, d\widetilde{B}_t \\ &\stackrel{mart}{=} U_{t-}\, dG_t + d[U,G]_t - \widetilde{B}_t\, dG_t - G_{t-}\, d\widetilde{B}_t \end{aligned}$$

Consequently,

$$d\big((U_t - \widetilde{B}_t)G_t\big) - (U_{t-} - \widetilde{B}_{t-})\, dG_t \stackrel{mart}{=} d[U,G]_t - d\langle U, M\rangle_u = d[U,B]_t + d[U,M]_t - d\langle U, M\rangle_u.$$

Hence the process given by (5.29) is a $(\mathbb{P},\mathbb{F})$-local martingale since, by Yœurp's lemma, the process $[U,B]_t$ is a $(\mathbb{P},\mathbb{F})$-local martingale and the process $[U,M] - \langle U,M\rangle$ is a $(\mathbb{P},\mathbb{F})$-local martingale as well. We conclude that the property (i) is valid in the present setup.

For (ii), we fix $u \geq 0$ and we observe that, for every $t \geq u$,

$$\widehat{U}^0_{u,t} = U_t - \widehat{B}_{u,t} - \widehat{U}_{u,u} = U_t - Z_{u,t} - V_u \tag{5.30}$$

where we denote $V_u = \widetilde{B}_u + \Delta U_u + \widehat{U}_{u,u}$ and $(Z_{u,t})_{t\geq u}$ is the $\mathbb{F}$-predictable process given by

$$Z_{u,t} = \int_{(u,t]} (F_{u,v-})^{-1}\, d\langle U, F_{u,\cdot}\rangle_v. \tag{5.31}$$

We fix $u, s \geq 0$. By applying the integration by parts formula, we obtain, for $t \geq s$,

$$d(F_{s,t}\widehat{U}^0_{u,t}) = U_{t-}\, dF_{s,t} + F_{s,t-}\, dU_t + d[U, F_{s,\cdot}]_t - Z_{u,t}\, dF_{s,t} - F_{s,t-}\, dZ_{u,t} - V_u\, dF_{s,t}$$

where we used the fact that the process $(Z_{u,t})_{t\geq u}$ is $\mathbb{F}$-predictable. Recall that U and $(F_{s,t})_{t\geq s}$ are $(\mathbb{P},\mathbb{F})$-martingales. Therefore, to show that the process $(F_{s,t}\widehat{U}^0_{u,t})_{t\geq u\vee s}$ is a $(\mathbb{P},\mathbb{F})$-local martingale, it is enough to check that, for all $u \leq s \leq t$ and $s \leq u \leq t$,

$$d[U, F_{s,\cdot}]_t - F_{s,t-}\, dZ_{u,t} \stackrel{\text{mart}}{=} d\left\langle U, F_{s,\cdot}\right\rangle_t - F_{s,t-}\, dZ_{u,t} \stackrel{\text{mart}}{=} 0. \tag{5.32}$$

Since we assumed that τ is a pseudo-honest time, using Remark 5.2.1, we obtain, for all $u \leq s \leq t$,

$$F_{s,t-}\, dZ_{u,t} = \frac{F_{s,t-}}{F_{u,t-}}\, d\left\langle U, F_{u,\cdot}\right\rangle_t = \frac{F_{s,t-}}{F_{u,t-}}\, d\left\langle U, \frac{F_{u,s}}{F_{s,s}}\, F_{s,\cdot}\right\rangle_t = \frac{F_{s,t-}F_{u,s}}{F_{u,t-}F_{s,s}}\, \left\langle U, F_{s,\cdot}\right\rangle_t = d\left\langle U, F_{s,\cdot}\right\rangle_t.$$

Similarly, for all $s \leq u \leq t$, we get

$$F_{s,t-}\, dZ_{u,t} = \frac{F_{s,t-}}{F_{u,t-}}\, d\left\langle U, F_{u,\cdot}\right\rangle_t = \frac{F_{s,u}F_{u,t-}}{F_{u,u}F_{u,t-}}\, d\left\langle U, F_{u,\cdot}\right\rangle_t = d\left\langle U, \frac{F_{s,u}}{F_{u,u}}\, F_{u,\cdot}\right\rangle_t = d\left\langle U, F_{s,\cdot}\right\rangle_t.$$

This shows that $d\left\langle U, F_{s,\cdot}\right\rangle_t - F_{s,t-}\, dZ_{u,t} = 0$ and thus the second equality (5.32) is trivially satisfied. This means, of course, that the property (ii) is satisfied. Using Theorem 5.4.1, we thus conclude that the process $\bar{U}$ given by (5.27) is a $(\mathbb{P},\mathbb{G})$-local martingale. This in turn implies that the process $\bar{U}$ given by (5.26) is a $(\mathbb{P},\mathbb{G})$-local martingale, as was required to show. □

The next result is borrowed from Kchia et al. [70] who examined the case of any two enlargements that coincide after a random time τ. However, for simplicity of presentation, their result (see Theorem 3 in [70]) is stated here for the special case of the progressive enlargement $\mathbb{G}$ and the initial enlargement $\mathbb{G}^*$, which are known to coincide after τ. For brevity, we write hereafter $\widetilde{C} = \langle U, M\rangle + \breve{U}^p$ where $\breve{U}^p$ is the dual $\mathbb{F}$-predictable projection of the process $\breve{U}_t = \Delta U_\tau \mathbb{1}_{\{\tau \le t\}}$.

Theorem 5.6.3. *Let U be a $(\mathbb{P},\mathbb{F})$-local martingale. Suppose that B is a $\mathbb{G}^*$-predictable process of finite variation such that $U - B$ is a $(\mathbb{P},\mathbb{G}^*)$-local martingale. Then the process*

$$U_t - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\widetilde{C}_t - \int_{(t\wedge\tau,t]} dB_u \tag{5.33}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale. Hence U is a $(\mathbb{P},\mathbb{G})$-special semimartingale.

Special Cases of Pseudo-Honest Times

From Theorem 5.6.2, we know that the hypothesis (H') holds for a pseudo-honest time with a strictly positive $(\mathbb{P},\mathbb{F})$-conditional distribution. Moreover, this result furnishes also a general expression for the canonical decomposition with respect to $\mathbb{G}$ of an arbitrary $(\mathbb{P},\mathbb{F})$-local martingale. Of course, any $\mathbb{F}$-predictable process of finite variation is also a $\mathbb{G}$-predictable process of finite variation and thus it suffices to focus on the canonical decomposition with respect to $\mathbb{G}$ of a $(\mathbb{P},\mathbb{F})$-local martingale, rather than a $(\mathbb{P},\mathbb{F})$-special semimartingale. Our next goal is to examine some useful consequences of Theorem 5.6.2 under alternative additional assumptions imposed on a pseudo-honest time under consideration. In particular, we will compare various semimartingale decompositions for pseudo-honest times with their classic counterparts established for honest times by Barlow [6], Jeulin and Yor [65], and Jeulin and Yor [66].

For the readers's convenience, we first recall the most pertinent results regarding the case of an honest time. It was shown by Barlow [6] (Theorem 3.10) and Yor [107] (Theorem 4) that the hypothesis (H') holds for $\mathbb{F}$ and its progressive enlargement with an honest time. The following result summarizes the well known properties of the progressive enlargement for an honest time (see Theorem A in Barlow [6], Theorem 2 in Jeulin and Yor [65], and Theorem 15 in Jeulin and Yor [66]).

Theorem 5.6.4. *Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ with an honest time τ. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then U is a $(\mathbb{P},\mathbb{G})$-special semimartingale.*
(i) The process

$$U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\widetilde{C}_u, \tag{5.34}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale.
(ii) The process

$$U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\langle U, \bar{M}\rangle_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\langle U, \bar{M}\rangle_u \tag{5.35}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale, where $\bar{M}$ is the $(\mathbb{P},\mathbb{F})$-martingale of class BMO introduced in part (ii) of Theorem 5.6.1.

Completely Separable Case

As a first special case of a pseudo-honest time, we consider the situation where the $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time τ is completely separable (see Definition 5.2.3). Then we obtain the following immediate corollary to Theorem 5.6.2. Corollary 5.6.1 will later be exemplified through the predictable multiplicative construction of a random time (see Corollary 5.6.3).

Corollary 5.6.1. *Under the assumptions of Theorem 5.6.2, we postulate, in addition, that the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is completely separable, that is, $F_{u,t} = K_u L_t$ for every $0 \le u \le t$*

where L is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process $\bar{U}$ is a $(\mathbb{P},\mathbb{G})$-local martingale where

$$\bar{U}_t = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u - \int_{(t\wedge\tau,t]} (L_{u-})^{-1}\, d\langle U, L\rangle_u. \tag{5.36}$$

Proof. We note that under the assumptions of Corollary 5.6.1, formula (5.27) reduces to (5.36). □

Predictable Multiplicative Construction

The next corollary shows that if τ is a pseudo-honest time with a non-degenerate $(\mathbb{P},\mathbb{F})$-conditional distribution then, under certain technical assumptions, the $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-local martingale is analogous to the one derived by other authors for an honest time and reported in Theorem 5.6.4. It is worth stressing that the present setup is manifestly different from the one covered by Theorem 5.6.4 and, in fact, our results do not cover the case of an honest time.

Let us observe that, under the assumptions of Theorem 5.6.2, for every $s \geq 0$, the process $(C_{s,u} = (F_u)^{-1}F_{s,u})_{u\geq s}$ is positive, decreasing, and $\mathbb{F}$-adapted. Indeed, from (5.3) we obtain the equality $(F_u)^{-1}F_{s,u} = (F_{u,t})^{-1}F_{s,t}$, which holds for all $0 \leq s < u < t$, where $(F_{u,t})_{u\geq 0}$ is an increasing process.

Corollary 5.6.2. *Let the assumptions of Theorem 5.6.2 be satisfied. Assume, in addition, that, for every $s \geq 0$, the decreasing process $(C_{s,u} = (F_u)^{-1}F_{s,u})_{u\geq s}$ is $\mathbb{F}$-predictable. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process $\bar{U}$ given by*

$$\bar{U}_t = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u + \int_{(t\wedge\tau,t]} ({}^pF_u)^{-1}\, d\langle U, M\rangle_u \tag{5.37}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale.

Proof. To show that the second integral in the right-hand side of (5.37) is a well-defined process of locally integrable variation, we observe that $0 < F_{u-} \leq {}^pF_u$ (since F is a submartingale) and thus

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left(\int_0^\infty \mathbb{1}_{\{\tau<u\}}({}^pF_u)^{-1}\, d|\langle U, M\rangle|_t\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty F_{u-}({}^pF_u)^{-1}\, d|\langle U, M\rangle_t|\right)\\ &\leq \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty d|\langle U, M\rangle_t|\right) \leq c\, \|U\|_{\mathcal{H}^1}\|M\|_{BMO}\end{aligned}$$

where we used Fefferman's inequality in the last inequality.

To obtain (5.37) from (5.26), we start by noting that $C_{s,u}$ is an $\mathbb{F}$-predictable multiplicative system associated with a positive submartingale F (see Meyer [85]) or [74]). By assumption, the process F_t is strictly positive for $t > 0$ and thus, by Theorem 4.1 and Corollary 4.1 in [74], the unique $\mathbb{F}$-predictable multiplicative system $C_{s,u}$ associated with F satisfies the following stochastic differential equations

$$dC_{s,u} = -C_{s,u-}({}^pF_u)^{-1}dA_u = -C_{s,u}(F_{u-})^{-1}dA_u \tag{5.38}$$

where the second equality follows from the equality $C_{s,u}{}^pF_u = C_{s,u-}F_{u-}$ (see formula (17) in [74]). Since the decreasing process $(C_{s,u})_{u\geq s}$ is assumed to be $\mathbb{F}$-predictable, the integration by parts formula yields, for any fixed $s \geq 0$,

$$dF_{s,u} = C_{s,u}\, dF_u + F_{u-}\, dC_{s,u} = -C_{s,u}\, dM_u$$

where the second equality follows from (5.38) and the Doob-Meyer decomposition $dF_t = dA_t - dM_t$. We only need to focus on the last term in formula (5.26). We have, for all $s \leq t$

$$\int_{(s,t]} (F_{s,u-})^{-1}\, d\langle U, F_{s,\cdot}\rangle_u = -\int_{(s,t]} (F_{u-}C_{s,u-})^{-1}C_{s,u}\, d\langle U, M\rangle_u = -\int_{(s,t]} ({}^pF_u)^{-1}\, d\langle U, M\rangle_u,$$

as was required to show. □

Remark 5.6.4. Let us consider the situation of Corollary 5.6.3 and let us assume, in addition, that the avoidance property (**A**) holds, that is, τ avoids all $\mathbb{F}$-stopping times. Then ${}^pF = F_-$ and the $(\mathbb{P},\mathbb{F})$-local martingale U is continuous at τ. Hence (5.37) becomes

$$\bar{U}_t = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\langle U, M\rangle_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\langle U, M\rangle_u \tag{5.39}$$

since, under assumption (**A**), the martingales M and $\bar{M}$ are known to coincide as well. Note that under assumption (**A**) alternative semimartingale decompositions (5.34) and (5.35) obtained for an honest time reduce to (5.37) as well.

We will now describe a particular instance where the assumptions of Corollary 5.6.3 are satisfied. For the reader's convenience, we will first summarize the main steps in an explicit construction of a random time associated with an arbitrary Azéma submartingale F, as developed in [74]. We now assume that we are given a predetermined Azéma submartingale F, that is, an arbitrary submartingale $F = (F_t)_{t\in\bar{\mathbb{R}}_+}$, defined on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, satisfying the inequalities $0 \le F_t \le 1$ for every $t \in \mathbb{R}_+$ and with $F_\infty = 1$. The predictable multiplicative construction of a random time τ associated with F runs as follows:

- We start by establishing the existence of an $\mathbb{F}$-predictable multiplicative system $\widehat{C}_{u,t}$ associated with a positive submartingale F (see Meyer [85] and Theorem 4.1 in [74]).
- Subsequently, using a (possibly non-unique) $\mathbb{F}$-predictable multiplicative system $\widehat{C}_{u,t}$, we define the unique $(\mathbb{P},\mathbb{F})$-conditional distribution $\widehat{F}_{u,t}$ by setting (see Theorem 4.2 and Lemma 5.1 in [74])
$$\widehat{F}_{u,t} = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(F_u \,|\, \mathcal{F}_t\right), & t\in[0,u),\\ \widehat{C}_{u,t}F_t, & t\in[u,\infty].\end{cases}$$
- Finally, we construct a random time τ on the extended probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{\mathbb{P}})$ such that $\widehat{\mathbb{P}}(\tau \le t \,|\, \mathcal{F}_t) = F_t$ for all $t\in\mathbb{R}_+$ (see Theorem 5.1 in [74]).

Recall that $\widehat{\mathbb{P}}$ is chosen in such a way that the probability measures $\widehat{\mathbb{P}}$ and $\mathbb{P}$ coincide on the filtration $\mathbb{F}$. It is also worth noting that if $G = M - A$ is the Doob-Meyer decomposition of G then, using the uniqueness of the Doob-Meyer decomposition, we deduce that $M_t = \mathbb{E}_{\widehat{\mathbb{P}}}(H^p_\infty \,|\, \mathcal{F}_t)$ and $A = H^p$. It is clear that Corollary 5.6.2 can now be applied to the random time τ constructed as above. Specifically, we are in a position to establish the following result, in which we use the fact that, under stronger assumptions on F, the unique $\mathbb{F}$-predictable multiplicative system $\widehat{C}_{u,t}$ associated with F is known explicitly (see [74]).

Corollary 5.6.3. *Assume that $F_t > 0$ and $F_{t-} > 0$ for every $t > 0$ and a pseudo-honest time τ is constructed using the unique $\mathbb{F}$-predictable multiplicative system associated with F. If U is a $(\widehat{\mathbb{P}},\mathbb{F})$-local martingale then the process $\bar{U}$ given by* (5.37) *is a $(\widehat{\mathbb{P}},\mathbb{G})$-local martingale.*

Proof. The statement follows immediately from Corollary 5.6.2. Alternatively, it can also be deduced from Corollary 5.6.1. To see this, we start by noting that Proposition 5.1 in [74] implies that $F_{u,t}$ is completely separable with $L = F\mathcal{E}$ where $\mathcal{E}$ is the Doléans exponential (see formula (30) in [74])

$$\mathcal{E}_t = \mathcal{E}_t\bigg(-\int_{(0,\,\cdot\,]} ({}^pF_s)^{-1}\, dA_s\bigg),$$

so that $d\mathcal{E}_t = -\mathcal{E}_{t-}({}^pF_t)^{-1}\,dA_t$. It is also known that $\mathcal{E}_t{}^pF_t = \mathcal{E}_{t-}F_{t-} = L_{t-}$ (this is a consequence of formula (17) in [74]). Since $\mathcal{E}$ is an $\mathbb{F}$-predictable process of finite variation, by applying the integration by parts formula, we obtain

$$dL_t = \mathcal{E}_t\, dF_t + F_{t-}\, d\mathcal{E}_t = ({}^pF_t)^{-1}L_{t-}\, dF_t - F_{t-}\mathcal{E}_{t-}({}^pF_t)^{-1}\, dA_t.$$

Consequently, we also have that

$$(L_{t-})^{-1}\,dL_t = -({}^pF_t)^{-1}\,dM_t + ({}^pF_t)^{-1}\,dA_t + (L_{t-})^{-1}F_{t-}\mathcal{E}_{t-}({}^pF_t)^{-1}\,dA_t = -({}^pF_t)^{-1}\,dM_t$$

and this in turn implies

$$(L_{t-})^{-1}\,d\langle U,L\rangle_t = -({}^pF_t)^{-1}\,d\langle U,M\rangle_t.$$

To conclude the proof, it suffices to apply Corollary 5.6.1. □

Remark 5.6.5. Corollary 5.6.3 corresponds to Theorem 7.1 in Jeanblanc and Song [59] who work under the assumption that $G_t = N_t e^{-\Lambda_t}$ where N is a positive local martingale and Λ is a continuous increasing process. Consequently, the martingale part M in the Doob-Meyer decomposition of G satisfies $dM_t = e^{-\Lambda_t}\,dN_t$. Moreover, they postulate that that $G_t < 1$ and $G_{t-} < 1$ for every $t > 0$. It is shown in [59] that one may construct a random time τ on the product space $\Omega \times \bar{\mathbb{R}}_+$ and with respect to a suitably defined probability measure $\mathbb{Q}$ such that the equality $\mathbb{Q} = \mathbb{P}$ is satisfied on $\mathbb{F}$ and $\mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = G_t$ for all $t \in \bar{\mathbb{R}}_+$. Jeanblanc and Song also show (see Theorem 7.1 in [59]) that if a process U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process $\bar{U}$ given by (5.39) is a $(\mathbb{P},\mathbb{G})$-local martingale. More precisely, under their assumptions, formula (5.39) becomes

$$\bar{U}_t = U_t - \int_{(0,t\wedge\tau]} \frac{e^{-\Lambda_u}}{G_{u-}}\,d\langle U,N\rangle_u + \int_{(t\wedge\tau,t]} \frac{e^{-\Lambda_u}}{F_{u-}}\,d\langle U,N\rangle_u.$$

For a more general result in this vein, see also Theorems 4.2 and 4.4 in Jeanblanc and Song [60]. It is worthwhile to observe that, in the setup considered in [59], the canonical solution τ satisfies the 'local density hypothesis in canonical form' or in another words, the hypothesis (ED) is satisfied (see Theorem 5.1 in [59]). This implies that for any $\mathbb{F}$-stopping time T (for the definition of $\mathcal{E}_t(u)$, see Corollary 3.1 in [59])

$$\mathbb{Q}(\tau = T \,|\, \mathcal{F}_\infty) = \int_{[\![T]\!]} N_u \mathcal{E}_\infty(u) e^{-\Lambda_u}\,d\Lambda_u = N_T \mathcal{E}_\infty(T) e^{-\Lambda_T}\Delta\Lambda_T = 0,$$

where we also used the assumption made in [59] that the process Λ is continuous. We conclude that the avoidance property (**A**) holds. As a consequence, the dual $\mathbb{F}$-predictable and $\mathbb{F}$-optional projections of H coincide and thus also $M = \bar{M}$. In our general setting, the assumption that Λ is continuous is equivalent to the assumption that the $\mathbb{F}$-predictable process A which generates G is continuous. Under this assumption, Theorem 7.1 in [59] can be recovered from Corollary 5.6.3.

Optional Multiplicative Construction

In the next special case, we assume that a pseudo-honest time τ is constructed using an $\mathbb{F}$-optional multiplicative system associated with a predetermined Azéma submartingale F. We suppose that we are given an Azéma submartingale F. In order to construct an $\mathbb{F}$-optional multiplicative system associated with F, we proceed as in Section 4.3 in [74]. Specifically, we start by assuming that F is given as $F_t = \mathbb{P}(\widehat{\tau} \le t \,|\, \mathcal{F}_t)$ for some random time $\widehat{\tau}$. Next, an $\mathbb{F}$-optional multiplicative system associated with F is defined. We may assume, without loss of generality, that an auxiliary random time $\widehat{\tau}$ was constructed using an $\mathbb{F}$-predictable multiplicative system $(\widehat{C}_{u,t})_{u,t\ge 0}$ associated with F. Hence this additional requirement is not restrictive.

More formally, to construct an $\mathbb{F}$-optional multiplicative system associated with F, we set $\widehat{H} = \mathbb{1}_{[\![\widehat{\tau},\infty [\![}$ and we define $\widehat{A} = \widehat{H}^o$ and $\widehat{M}_t = \mathbb{E}_{\mathbb{P}}(\widehat{H}^o_\infty \,|\, \mathcal{F}_t)$. As in Remark 5.6.1, we note that the equality $G = \widehat{M} - \widehat{A}$ yields an $\mathbb{F}$-optional decomposition of G. We define the random field $(C_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ by setting $C_{u,t} = 1$ for all $u \ge t$ and, for all $t \ge u$,

$$dC_{u,t} = -C_{u,t-}(F_t)^{-1}\,d\widehat{A}_t. \tag{5.40}$$

Then, from Corollary 4.2 in [74], the random field $(C_{s,u})_{s,u\in\bar{\mathbb{R}}_+}$ is an $\mathbb{F}$-optional multiplicative system associated with F. The $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time is now defined by

$$F_{u,t} = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(F_u \,\middle|\, \mathcal{F}_t\right), & t \in [0,u), \\ C_{u,t}F_t, & t \in [u,\infty]. \end{cases}$$

Finally, the random time τ can be constructed using once again Theorem 5.1 in [74]. It is then not difficult to check that τ is a pseudo-honest time. It is important to emphasize that we do not claim that the equality $\widehat{A} = H^o$ holds where, as usual, we write $H = \mathbb{1}_{[\![\tau,\infty[\![}$. Therefore, the $\mathbb{F}$-optional decomposition of $G = \bar{M} - \bar{A}$, which is obtained for a random time τ as outlined in Remark 5.6.1, does not coincide with the $\mathbb{F}$-optional decomposition $G = \widehat{M} - \widehat{A}$, which is associated with an auxiliary random time $\widehat{\tau}$.

Corollary 5.6.4. *Let the assumptions of Theorem 5.6.2 be satisfied by a pseudo-honest time τ constructed using the $\mathbb{F}$-optional multiplicative system given by* (5.40)*. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process*

$$U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\langle U,\bar{M}\rangle_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\langle U,\widehat{M}\rangle_u \tag{5.41}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale.

Proof. The first integral in (5.41) is dealt with as in part (ii) of Theorem 5.6.1. For the second integral in (5.41), we start by noting that it is a well-defined process of locally integrable variation, since

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(\int_0^\infty \mathbb{1}_{\{\tau<t\}}(F_{t-})^{-1}\, d|\langle U,\widehat{M}\rangle|_t\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty F_{t-}(F_{t-})^{-1}\, d|\langle U,\widehat{M}\rangle|_t\right) \\
&\leq \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty d|\langle U,\widehat{M}\rangle|_t\right) \leq c\,\|U\|_{\mathcal{H}^1}\|\widehat{M}\|_{BMO}
\end{aligned}$$

where the local martingale U is locally in $\mathcal{H}^1$ and $\widehat{M}$ is the BMO martingale. Since the process $(C_{s,u})_{u\geq s}$ is decreasing, the integration by parts formula yields, for any fixed $s \geq 0$,

$$dF_{s,u} = C_{s,u-}\, dF_u + F_u\, dC_{s,u} = -C_{s,u-}\, d\widehat{M}_u$$

where the second equality follows from (5.40) and the decomposition $G = \widehat{M} - \widehat{A}$. Hence, for all $s \leq t$,

$$\int_{(s,t]} (F_{s,u-})^{-1}\, d\langle U,F_{s,\cdot}\rangle_u = -\int_{(s,t]} (F_{u-}C_{s,u-})^{-1}C_{s,u-}\, d\langle U,\widehat{M}\rangle_u = -\int_{(s,t]} (F_{u-})^{-1}\, d\langle U,\widehat{M}\rangle_u.$$

To conclude the proof, we combine Theorem 5.6.1 with Theorem 5.6.2. □

5.6.2 Hypothesis (H') for Pseudo-Initial Times

In this subsection, we examine the case of a pseudo-initial time. The first result is that useful result showing that the hypothesis (H') is met, that is, any $(\mathbb{P},\mathbb{F})$-semimartingale is also a $(\mathbb{P},\mathbb{G})$-semimartingale. Note that Theorem 5.6.5 is an extension of the classic result due to Jacod [50], who dealt with the case of an initial time (i.e., the density hypothesis) and the initial enlargement $\mathbb{G}^*$.

Theorem 5.6.5. *If τ is a pseudo-initial time then the hypothesis (H') is satisfied by $\mathbb{F}$ and and the progressive enlargement $\mathbb{G}$.*

Proof. Given a $\mathbb{F}$-semimartingale X, we will prove the claim by contradiction. Suppose X is not a $\mathbb{G}$-semimartingale, then there exists $t \geq 0$, $\epsilon > 0$ and a sequence of simple $\mathbb{G}$-predictable process $\{\xi^n\}_{n\in\mathbb{N}}$ converging uniformly to zero, such that

$$\inf_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left(1 \wedge \left|(\xi^n \bullet X)_t\right|\right) \geq \epsilon \tag{5.42}$$

and for large enough n, the process $(\xi^n_s)_{s\geq 0}$ is bounded $\mathbb{F}^\tau$-predictable.

The $\mathbb{F}$-conditional distribution of τ is absolutely continuous with respect to a $\mathbb{F}$-adapted non-decreasing process. Therefore we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(1 \wedge \left|(\xi^n \bullet X)_t\right|\right) &= \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty[} \min\left(1, \left|\int_{(0,t]} \xi^n_s(u\wedge s)\, dX_s\right|\right) F_t(du)\right)\\
&= \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty[} \min\left(1, \left|\int_{(0,t]} \xi^n_s(u\wedge s)\, dX_s\right|\right) m_{u,t}\, dD_u\right)\\
&= \int_{[0,\infty[} \mathbb{E}_{\mathbb{P}}\left(\min\left(1, \left|\int_{(0,t]} \xi^n_s(C_u\wedge s)\, dX_s\right|\right) m_{C_u,t}\mathbb{1}_{\{C_u<\infty\}}\right) du
\end{aligned}$$

The process $C_u \wedge s$ is continuous in s and is $\mathbb{F}$-adapted, since for any $k \in \mathbb{R}_+$,

$$\{C_u \wedge s > k\} = \{C_u > k\} \cap \{s > k\} = \begin{cases} \{\phi\} \in \mathcal{F}_s & s \leq k \\ \{C_u > k\} \in \mathcal{F}_k \subseteq \mathcal{F}_s & s > k \end{cases}$$

This implies the map $(s,\omega) \to C_u(\omega) \wedge s$ is $\mathbb{F}$-predictable and therefore $\sigma(C_u \wedge \cdot) \subseteq \mathcal{P}(\mathbb{F})$.

By assumption, for every $n \in \mathbb{N}$ and $u \geq 0$, the process $(\xi^n_s(u))_{s\geq 0}$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\bar{\mathbb{R}}_+)$ measurable. This implies the map

$$(s,\omega) \longrightarrow \xi^n_s(C_u(\omega)\wedge s)(\omega)$$

is $\mathcal{P}(\mathbb{F}) \wedge \sigma(C_u \wedge \cdot)$ measurable. We have just shown that $\sigma(C_u \wedge \cdot) \subseteq \mathcal{P}(\mathbb{F})$ and this implies for all $n \in \mathbb{N}$, the process $(\xi^n_s(C_u \wedge s))_{s\geq 0}$ is $\mathbb{F}$-predictable.

By an application of dominated convergence theorem, We only need to show that the intergrand converges to zero. That is

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left(\min\left(1, \left|\int_{(0,t]} \xi^n_s(C_u\wedge s)\, dX_s\right|\right) m_{C_u,t}\mathbb{1}_{\{C_u<\infty\}}\right) = 0.$$

To see this, one first define a measure $\mathbb{Q}_t$ by $d\mathbb{Q}_t = m_{C_u,t}\mathbb{1}_{\{C_u<\infty\}}d\mathbb{P}$ and then we apply Theorem 2.1.1 with the fact that the class of semimartingale is invariant under an absolute change of measure. □

We will now derive the $\mathbb{G}$-semimartingale decomposition for a progressive enlargement with a pseudo-initial time. Note that the $\mathbb{G}$-semimartingale decomposition established in the literature under the density hypothesis by Jeanblanc and Le Cam [57] (see also Kchia et al. [70] who employed their Theorem 5.6.3) can be obtained as a special case of equality (5.43) by postulating that the increasing process D in Definition 5.2.5 is non-random. Our proof of decomposition (5.43) is based on Theorem 5.6.2.

Theorem 5.6.6. *Let τ be a pseudo-initial time. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process $\bar{U}$ is a $(\mathbb{P},\mathbb{G})$-local martingale where*

$$\bar{U}_t = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u + \int_{(t\wedge s,t]} (m_{s,u-})^{-1}\, d\langle U, m_{s,\cdot}\rangle_u \big|_{s=\tau}. \tag{5.43}$$

Proof. We maintain the notation introduced in the proof of Theorem 5.6.2. In view of Proposition 5.4.1 and the arguments used in the proof of Theorem 5.6.2, it suffices to show that, for any fixed $u \geq 0$, the process $(m_{u,t}\widehat{U}^0_{u,t})_{t\geq u}$ is a $(\mathbb{P},\mathbb{F})$-local martingale, where we set $\widehat{U}^0_{u,t} = U_t - Z_{u,t} - V_u$ (see (5.30)), where in turn $V_u = \widetilde{B}_u + \Delta U_u + \widehat{U}_{u,u}$, with the process $\widetilde{B}$ defined by (5.28) and $Z_{u,t}$ given by the following expression (see (5.31))

$$Z_{u,t} = \int_{(u,t]} (m_{u,v-})^{-1}\, d\langle U, m_{u,\cdot}\rangle_v .$$

By applying the integration by parts formula, we obtain

$$\begin{aligned} d(m_{u,t}\widehat{U}^0_{u,t}) &= U_{t-}\, dm_{s,t} + m_{s,t-}\, dU_t + d\langle U, m_{s,\cdot}\rangle_t - Z_{u,t-}\, dm_{s,t} - m_{u,t}\, dZ_{u,t} - V_u\, dm_{u,t} \\ &= U_{t-}\, dm_{s,t} + m_{s,t-}\, dU_t - Z_{u,t-}\, dm_{u,t} - V_u\, dm_{u,t}, \end{aligned}$$

which is clearly a $(\mathbb{P},\mathbb{F})$-local martingale for $t \geq u$. □

Let us assume, in addition, that there exists a positive $(\mathbb{P},\mathbb{F})$-martingale L and a positive, $\mathbb{F}$-adapted process a such that the equality $m_{s,t} = L_t a_s$ holds for all $s \leq t$. Then

$$F_{u,t} = \int_{[0,u]} m_{s,t}\, dD_s = L_t \int_{[0,u]} a_s\, dD_s = K_u L_t$$

where $K_u = \int_{[0,u]} a_s\, dD_s$ and thus the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is completely separable. Then formula (5.43) can also be deduced from Corollary 5.6.1.

Corollary 5.6.5. *Let τ be a pseudo-initial time. Assume that there exists a positive $(\mathbb{P},\mathbb{F})$-martingale L and a positive, $\mathbb{F}$-adapted process a such that $m_{s,t} = L_t a_s$ for all $s \leq t$. Then the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is completely separable. Moreover, if U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process $\bar{U}$ given by (5.43) is a $(\mathbb{P},\mathbb{G})$-local martingale.*

Proof. It suffices to apply Corollary 5.6.1 and observe that, for any fixed s, the equality $m_{s,u} = a_s L_u$ holds. Hence formulae (5.36) and (5.43) are equivalent. □

5.7 Applications to Financial Mathematics

In this final section, we present two applications of some of our general results established in preceding sections to specific problems arising in the context of financial modeling.

5.7.1 Arbitrage Free Markets Models

In the recent paper by Coculescu et al. [27], the existence of an equivalent probability measure under which the immersion property holds was shown to be a sufficient condition for a market model with enlarged filtration to be arbitrage-free, provided that the underlying market model based on the filtration $\mathbb{F}$ enjoyed this property. In Proposition 5.7.3, we will show that if τ satisfies the hypothesis (*HP*) (or, more precisely, the complete separability of the conditional distribution $F_{u,t}$ holds) then, under mild technical assumptions, the result from [27] can be applied to the progressive enlargement $\mathbb{G}$.

Immersion Property under an Equivalent Probability Measure

Before studying the case of a progressive enlargement, we first summarize briefly some results from Coculescu et al. [27] and make pertinent comments. Suppose that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with arbitrary filtrations $\mathbb{F}$ and $\mathbb{K}$ such that $\mathbb{F} \subset \mathbb{K}$. Let $\mathcal{I}(\mathbb{P})$ stand for the class of all probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{F}$ and $\mathbb{K}$ satisfy the immersion property under $\mathbb{Q}$. Recall that the immersion property for $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{Q}$ stipulates that any $(\mathbb{Q}, \mathbb{F})$-local martingale is a $(\mathbb{Q}, \mathbb{K})$-local martingale. The following lemma was established in [27].

Lemma 5.7.1. *Assume that $\widetilde{\mathbb{Q}} \in \mathcal{I}(\mathbb{P})$ and $\mathbb{Q}$ is a probability measure equivalent to $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F})$ such that $\frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}}$ is $\mathcal{F}_\infty$-measurable. Then $\mathbb{Q}$ belongs to $\mathcal{I}(\mathbb{P})$.*

Proof. Let M be an $(\mathbb{F}, \mathbb{Q})$-martingale. We wish to show that M is also a $(\mathbb{Q}, \mathbb{G})$-martingale. This is equivalent to the property that $M\eta$ is a $(\widetilde{\mathbb{Q}}, \mathbb{G})$-martingale where $\eta_t := \frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}}|_{\mathcal{G}_t}$. Since M and η are $\mathbb{F}$-adapted processes and $\widetilde{\mathbb{Q}} \in \mathcal{I}(\mathbb{P})$, it suffices to show that $M\eta$ is a $(\widetilde{\mathbb{Q}}, \mathbb{F})$-martingale. To this end, we observe that η is an $\mathbb{F}$-adapted $(\widetilde{\mathbb{Q}}, \mathbb{G})$-martingale and thus an $(\widetilde{\mathbb{Q}}, \mathbb{F})$-martingale. Hence the Bayes formula yields, for all $0 \leq t < s$,

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}(M_s \eta_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{F}_t)\, \mathbb{E}_{\widetilde{\mathbb{Q}}}(\eta_s \,|\, \mathcal{F}_t) = M_t \eta_t,$$

so that M is a $(\mathbb{Q}, \mathbb{G})$-martingale. By the usual localization argument, the proof can be extended to local martingales. □

Lemma 5.7.2. *Assume that the class $\mathcal{I}(\mathbb{P})$ is non-empty. Then for every $\widetilde{\mathbb{Q}} \in \mathcal{I}(\mathbb{P})$ there exists a probability measure $\mathbb{Q}$ equivalent to $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F})$ and such that the following conditions are met:*
(i) the Radon-Nikodým density process $\eta_t := \frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}}|_{\mathcal{G}_t}$ is $\mathbb{F}$-adapted,
(ii) the probability measures $\mathbb{Q}$ and $\mathbb{P}$ coincide on $\mathbb{F}$,
(iii) $\mathbb{Q}$ belongs to $\mathcal{I}(\mathbb{P})$,
(iv) every $(\mathbb{P}, \mathbb{F})$-local martingale is a $(\mathbb{Q}, \mathbb{K})$-local martingale.

Proof. (i) Let $\rho_\infty = d\mathbb{P}/d\widetilde{\mathbb{Q}}$ and let $\eta_\infty = \mathbb{E}_{\widetilde{\mathbb{Q}}}(\rho_\infty \,|\, \mathcal{F}_\infty)$. We define the probability measure $\mathbb{Q}$ equivalent to $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{G})$ by setting

$$\frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}} = \eta_\infty. \tag{5.44}$$

Then

$$\eta_t := \frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}}\Big|_{\mathcal{G}_t} = \mathbb{E}_{\widetilde{\mathbb{Q}}}(\eta_\infty \,|\, \mathcal{G}_t) = \mathbb{E}_{\widetilde{\mathbb{Q}}}(\eta_\infty \,|\, \mathcal{F}_t) = \mathbb{E}_{\widetilde{\mathbb{Q}}}(\rho_\infty \,|\, \mathcal{F}_t) = \frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}}\Big|_{\mathcal{F}_t}$$

where the third equality holds since $\widetilde{\mathbb{Q}} \in \mathcal{I}(\mathbb{P})$ and η_∞ is $\mathcal{F}_\infty$-measurable. We thus see that (i) holds.
(ii) Furthermore,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}}|_{\mathcal{F}_t}}{\frac{d\mathbb{P}}{d\widetilde{\mathbb{Q}}}|_{\mathcal{F}_t}} = \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}(\rho_\infty \,|\, \mathcal{F}_t)}{\mathbb{E}_{\widetilde{\mathbb{Q}}}(\rho_\infty \,|\, \mathcal{F}_t)} = 1$$

and thus (ii) is satisfied.
(iii) The third statement follows from (5.44) and Lemma 5.7.1. It is also worth noting that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_t} = \frac{\frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}}|_{\mathcal{G}_t}}{\frac{d\mathbb{P}}{d\widetilde{\mathbb{Q}}}|_{\mathcal{G}_t}} = \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}(\rho_\infty \,|\, \mathcal{F}_t)}{\mathbb{E}_{\widetilde{\mathbb{Q}}}(\rho_\infty \,|\, \mathcal{G}_t)},$$

where the right-hand side can be checked to be a $(\mathbb{P}, \mathbb{K})$-martingale.
(iv) Since $\mathbb{Q} = \mathbb{P}$ on $\mathbb{F}$, any $(\mathbb{P}, \mathbb{F})$-local martingale is a $(\mathbb{Q}, \mathbb{F})$-local martingale. From part (iii), we know that $\mathbb{Q} \in \mathcal{I}(\mathbb{P})$ and thus any $(\mathbb{Q}, \mathbb{F})$-martingale is also a $(\mathbb{Q}, \mathbb{K})$-martingale. Hence any $(\mathbb{P}, \mathbb{F})$-local martingale is also a $(\mathbb{Q}, \mathbb{K})$-martingale. □

Note that the probability measure $\mathbb{P}$ in Lemma 5.7.2 can be replaced by any probability measure equivalent to $\mathbb{P}$. This means that the assumption that the class $\mathcal{I}(\mathbb{P})$ is non-empty is fairly strong; it implies that for any probability measure $\mathbb{P}'$ there exists a probability measure $\mathbb{Q}'$ equivalent to $\mathbb{P}'$ on

$(\Omega, \mathcal{F})$, coinciding with $\mathbb{P}'$ on $\mathbb{F}$, and such that the immersion property between $\mathbb{F}$ and $\mathbb{K}$ holds under $\mathbb{Q}'$. It is thus natural to ask under which (non-trivial) circumstances the class $\mathcal{I}(\mathbb{P})$ is non-empty. A partial answer to this question for the progressive enlargement is provided in Proposition 5.7.3.

Remark 5.7.1. The property that the class $\mathcal{I}(\mathbb{P})$ is non-empty is not satisfied by $\mathbb{F}$ and the progressive enlargement $\mathbb{G}$ when τ is an $\mathcal{F}_\infty$-measurable random time (e.g., an honest time with respect to $\mathbb{F}$), unless it is an $\mathbb{F}$-stopping time (so that $\mathbb{F} = \mathbb{G}$ and the immersion property is trivially satisfied).

The next result, also borrowed from [27], provides a complete characterization of non-emptiness of the class $\mathcal{I}(\mathbb{P})$.

Proposition 5.7.1. *The following conditions are equivalent:*
(i) the class $\mathcal{I}(\mathbb{P})$ is non-empty,
(ii) there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that every $(\mathbb{P}, \mathbb{F})$-local martingale is a $(\mathbb{Q}, \mathbb{K})$-local martingale.

Proof. (i) $\Rightarrow$ (ii) This implication is an immediate consequence of part (iv) in Lemma 5.7.2.

(ii) $\Rightarrow$ (i) Assume that there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ such that any $(\mathbb{F}, \mathbb{P})$-local martingale is a $(\mathbb{Q}, \mathbb{K})$-local martingale. We claim that the hypothesis (H) holds under $\mathbb{Q}$. To this end, it suffices to show that $\mathbb{Q} = \mathbb{P}$ on $\mathbb{F}$. Then any $(\mathbb{Q}, \mathbb{F})$-local martingale is obviously an $(\mathbb{P}, \mathbb{F})$-local martingale and thus, by assumption, it is also a $(\mathbb{Q}, \mathbb{K})$-local martingale. This means that the hypothesis (H) holds under $\mathbb{Q}$.

It remains to show that (ii) implies that $\mathbb{Q} = \mathbb{P}$ on $\mathbb{F}$. For this purpose, we note that the process $\eta_t := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ is an $(\mathbb{P}, \mathbb{F})$-martingale. Hence, by assumption, it is also a $(\mathbb{Q}, \mathbb{K})$-martingale. Since η is an $\mathbb{F}$-adapted process, we see that it is a $(\mathbb{Q}, \mathbb{F})$-martingale. By the usual argument (see Lemma 5.7.1), this in turn is equivalent to the property that η^2 is an $(\mathbb{P}, \mathbb{F})$-local martingale. Since η and η^2 are $(\mathbb{P}, \mathbb{F})$-martingales, we conclude that η is constant process: $\eta = 1$, and thus the probability measures $\mathbb{Q}$ and $\mathbb{P}$ coincide on $\mathbb{F}$. $\square$

Martingale Measures via the Hypothesis (H)

Suppose now we are given a $(\mathbb{P}, \mathbb{F})$-semimartingale X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{M}(\mathbb{P}, \mathbb{F})$ stand for the class of all $\mathbb{F}$-*local martingale measures* for X, meaning that a probability measure $\mathbb{Q}$ belongs to $\mathcal{M}(\mathbb{P}, \mathbb{F})$ whenever (i) $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and (ii) X is an $(\mathbb{Q}, \mathbb{F})$-local martingale. Let $\mathbb{K}$ be any enlargement of the filtration $\mathbb{F}$. We denote by $\mathcal{M}(\mathbb{P}, \mathbb{K})$ the class of $\mathbb{K}$-*local martingale measures* for X. In [27], the authors assumed that the class $\mathcal{M}(\mathbb{P}, \mathbb{F})$ is non-empty and they searched for sufficient conditions ensuring that the class $\mathcal{M}(\mathbb{P}, \mathbb{K})$ is non-empty as well.

One possibility is to postulate that the immersion property holds under some probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and to infer that it is also valid under some $\mathbb{F}$-local martingale measure. Obviously, any $\mathbb{F}$-local martingale measure under which the immersion property holds is also a $\mathbb{K}$-local martingale measure. A particular example of an $\mathbb{F}$-local martingale measure under which the the immersion property holds can be produced using Lemma 5.7.2, leading to the following result, also due to Coculescu et al. [27] (see Corollary 4.6 therein).

Proposition 5.7.2. *(i) The classes $\mathcal{M}(\mathbb{P}, \mathbb{F})$ and $\mathcal{I}(\mathbb{P})$ are non-empty if and only if the set $\mathcal{M}(\mathbb{P}, \mathbb{F}) \cap \mathcal{I}(\mathbb{P})$ is non-empty.*
(ii) If the classes $\mathcal{M}(\mathbb{P}, \mathbb{F})$ and $\mathcal{I}(\mathbb{P})$ are non-empty then the class $\mathcal{M}(\mathbb{P}, \mathbb{K})$ is non-empty.

Proof. (i) Assume that $\mathcal{M}(\mathbb{P}, \mathbb{F})$ and $\mathcal{I}(\mathbb{P})$ are non-empty. Let $\widetilde{\mathbb{Q}} \in \mathcal{I}(\mathbb{P})$ and let $\mathbb{P}' \in \mathcal{M}(\mathbb{P}, \mathbb{F})$. Let $\mathbb{Q}$ be defined as in Lemma 5.7.2 with $\mathbb{P}$ replaced by $\mathbb{P}'$ (of course, $\mathcal{I}(\mathbb{P}) = \mathcal{I}(\mathbb{P}')$ since $\mathbb{P}$ is equivalent to $\mathbb{P}'$). Then $\mathbb{Q} \in \mathcal{I}(\mathbb{P})$ by part (iii) in Lemma 5.7.2. Also, by part (ii) in Lemma 5.7.2, the probability measures $\mathbb{Q}$ and $\mathbb{P}'$ coincide on $\mathbb{F}$ and thus $\mathbb{Q} \in \mathcal{M}(\mathbb{P}, \mathbb{F})$. Hence $\mathbb{Q} \in \mathcal{M}(\mathbb{P}, \mathbb{F}) \cap \mathcal{I}(\mathbb{P})$ and thus the class $\mathcal{M}(\mathbb{P}, \mathbb{F}) \cap \mathcal{I}(\mathbb{P})$ is non-empty. The converse implication is trivial.

(ii) In view of part (i), it suffices to show that any probability measure $\mathbb{Q}$ belonging to $\mathcal{M}(\mathbb{P},\mathbb{F})\cap\mathcal{I}(\mathbb{P})$ is in $\mathcal{M}(\mathbb{K},\mathbb{P})$. Indeed, any $(\mathbb{Q},\mathbb{F})$-local martingale is a $(\mathbb{Q},\mathbb{K})$-local martingale (since the immersion property holds under $\mathbb{Q}$) and X is an $(\mathbb{Q},\mathbb{F})$-local martingale (since $\mathbb{Q}$ is an $\mathbb{F}$-local martingale measure), so that X is a $(\mathbb{Q},\mathbb{K})$-local martingale, as required. □

Remark 5.7.2. An inspection of the proof of Proposition 5.7.2 (i.e. Corollary 4.6 in [27]) shows that the process X plays no essential role (except, of course, for the assumption that the class $\mathcal{M}(\mathbb{P},\mathbb{F})$ is non-empty). Note also that the probability measure $\mathbb{Q}$ constructed in this proof has the property that every $(\mathbb{P}',\mathbb{F})$-local martingale is also an $(\mathbb{Q},\mathbb{F})$-local martingale (since $\mathbb{P}'$ and $\mathbb{Q}$ coincide on $\mathbb{F}$) and thus a $(\mathbb{Q},\mathbb{K})$-local martingale (since $\mathbb{Q}\in\mathcal{I}(\mathbb{P}')$).

Remark 5.7.3. The class $\mathcal{M}(\mathbb{P},\mathbb{F})$ can be replaced in part (i) of Proposition 5.7.2 by any subset $\mathcal{P}$ of probability measures equivalent to $\mathbb{P}$ and such that: if $\mathbb{P}'\in\mathcal{P}$ and $\mathbb{P}''=\mathbb{P}'$ on $\mathbb{F}$ then $\mathbb{P}''\in\mathcal{P}$.

To summarize the conclusions from Coculescu et al. [27], the following conditions are equivalent:
(1) $\mathcal{M}(\mathbb{P},\mathbb{F})$ and $\mathcal{I}(\mathbb{P})$ are non-empty,
(2) $\mathcal{M}(\mathbb{P},\mathbb{F})\cap\mathcal{I}(\mathbb{P})$ is non-empty,
(3) $\mathcal{M}(\mathbb{P},\mathbb{F})$ is non-empty and there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F})$ such that every $(\mathbb{P},\mathbb{F})$-local martingale is a $(\mathbb{Q},\mathbb{K})$-local martingale.

In view of Proposition 5.7.2, any of conditions (1)–(3) implies that the class $\mathcal{M}(\mathbb{P},\mathbb{K})$ is non-empty. It is thus natural to refer to any of conditions (1)–(3) is the *no-arbitrage condition* for the market model with an enlarged filtration $\mathbb{K}$.

Martingale Measures for the Progressive Enlargement

We now consider the case where $\mathbb{K}=\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$. Suppose that the hypothesis (H) is not satisfied by the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ of a random time τ. It is then natural to ask whether there exists a probability measure $\bar{\mathbb{P}}$, which is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F})$ and such that the $(\bar{\mathbb{P}},\mathbb{F})$-conditional distribution $\bar{F}_{u,t}$ of τ satisfies the hypothesis (H). Equivalently, we ask whether there exists a probability measure $\bar{\mathbb{P}}$ equivalent to $\mathbb{P}$ and such that, for all $0\le u\le t$,

$$\bar{F}_{u,u}:=\bar{\mathbb{P}}\left(\tau\le u\,\middle|\,\mathcal{F}_u\right)=\bar{\mathbb{P}}\left(\tau\le u\,\middle|\,\mathcal{F}_t\right)=:\bar{F}_{u,t}. \tag{5.45}$$

Under the density hypothesis, the answer to this question is known to be positive (see El Karoui et al. [36] and Grorud and Pontier [46]). By contrast, when τ is assumed to be an honest time then this property never holds, unless τ is an $\mathbb{F}$-stopping time (see Remark 5.7.1).

In this subsection, we work under the standing assumption that the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is separable and $F_0=0$, so that τ is a pseudo-honest time (see Remark 5.2.2). Note, however, that the case of the classic honest time is not covered by foregoing results, since we also assume from now on that $F_{u,t}>0$ for all $0<u\le t$. Recall also that the complete separability implies that τ is a pseudo-honest time.

Lemma 5.7.3. *Assume that $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is completely separable, so that $F_{u,t}=K_uL_t$ for $0\le u\le t$. Let the process $(Z^{\mathbb{G}}_t)_{t\ge 0}$ be given by*

$$Z^{\mathbb{G}}_t=\widetilde{Z}_t\mathbb{1}_{\{\tau>t\}}+\widehat{Z}_{\tau,t}\mathbb{1}_{\{\tau\le t\}} \tag{5.46}$$

where $\widehat{Z}_{u,t}=\frac{F_{u,u}}{F_{u,t}}$ and

$$\widetilde{Z}_t=(G_t)^{-1}\bigg(1-\int_{(0,t]}\widehat{Z}_{u,t}\,dF_{u,t}\bigg)=(G_t)^{-1}\bigg(1-\mathbb{E}_{\mathbb{P}}\left(\widehat{Z}_{\tau,t}\mathbb{1}_{\{\tau\le t\}}\,\middle|\,\mathcal{F}_t\right)\bigg).$$

Then the process $Z^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$-local martingale.

Proof. The proof relies on an application of Corollary 5.4.2 and Theorem 5.4.1. First, $\widehat{Z}_{u,u} = 1$ and thus it is trivially an $\mathbb{F}$-predictable process. Next, we need to check condition (i) in Corollary 5.4.2, which now reads: the process $(W_t)_{t\geq 0}$ is a $(\mathbb{P},\mathbb{F})$-local martingale where

$$W_t = \widetilde{Z}_t G_t + \int_{(0,t]} \widehat{Z}_{u,u}\, dF_u = \widetilde{Z}_t G_t + F_t. \tag{5.47}$$

We observe that

$$\begin{aligned}\widetilde{Z}_t G_t + F_t &= 1 + F_t - \int_{(0,t]} \widehat{Z}_{u,t}\, dF_{u,t} \\ &= 1 + L_t K_t - \int_{(0,t]} L_u\, dK_u = 1 + L_0 K_0 + \int_{(0,t]} K_{u-}\, dL_u,\end{aligned}$$

so that the process W is indeed a $(\mathbb{P},\mathbb{F})$-local martingale. Finally, we need to check condition (ii) in Corollary 5.4.2, which takes here the following form: for every $u > 0$, the process $(W_{u,t} = L_t(\widehat{Z}_{u,t} - \widehat{Z}_{u,u}))_{t\geq u}$ is a $(\mathbb{P},\mathbb{F})$-local martingale. To this end, we note that

$$W_{u,t} = L_t \frac{F_{u,u}}{F_{u,t}} - L_t \widehat{Z}_{u,u} = L_t \frac{K_u L_u}{K_u L_t} - L_t \widehat{Z}_{u,u} = L_u - L_t \widehat{Z}_{u,u}, \tag{5.48}$$

which is, obviously, a $(\mathbb{P},\mathbb{F})$-local martingale for $t \geq u$. In view of Theorem 5.4.1, we conclude that $Z^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$-local martingale. □

We will also need the following simple lemma.

Lemma 5.7.4. *Assume that τ is a pseudo-honest time. Then for any $\mathbb{F}$-adapted, $\mathbb{P}$-integrable process X we have that, for every $s \leq t$,*

$$F_{s,t}\, \mathbb{E}_{\mathbb{P}}(X_\tau \mathbb{1}_{\{\tau\leq s\}} \,|\, \mathcal{F}_s) = F_{s,s}\, \mathbb{E}_{\mathbb{P}}(X_\tau \mathbb{1}_{\{\tau\leq s\}} \,|\, \mathcal{F}_t). \tag{5.49}$$

Proof. Note that for $X = 1$ equality (5.49) is trivially satisfied. In general, it suffices to consider an elementary $\mathbb{F}$-adapted process of the form $X_t = \mathbb{1}_A \mathbb{1}_{[u,\infty)}(t)$ for a fixed, but arbitrary, $u \geq 0$ and any event $A \in \mathcal{F}_u$. Obviously, both sides of (5.49) vanish when $u > s$. For any $u \leq s$, we obtain

$$\begin{aligned}&F_{s,t}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A \mathbb{1}_{[u,\infty)}(\tau) \mathbb{1}_{\{\tau\leq s\}} \,|\, \mathcal{F}_s) = \mathbb{1}_A F_{s,t}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq s\}} - \mathbb{1}_{\{\tau<u\}} \,|\, \mathcal{F}_s) = \mathbb{1}_A F_{s,t}(F_{s,s} - F_{u-,s}) \\ &= \mathbb{1}_A (F_{s,t}F_{s,s} - F_{u-,t}F_{s,s}) = \mathbb{1}_A F_{s,s}(F_{s,t} - F_{u-,t}) = \mathbb{1}_A F_{s,s}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq s\}} - \mathbb{1}_{\{\tau<u\}} \,|\, \mathcal{F}_t) \\ &= F_{s,s}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A \mathbb{1}_{[u,\infty)}(\tau) \mathbb{1}_{\{\tau\leq s\}} \,|\, \mathcal{F}_t)\end{aligned}$$

where we used condition (5.3) in the third equality. □

Proposition 5.7.3. *Assume that:*
(i) the $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time τ is completely separable and $F_0 = 0$,
(ii) the process $Z^{\mathbb{G}}$ given by formula (5.46) is a positive $(\mathbb{P},\mathbb{G})$-martingale such that $\mathbb{E}_{\mathbb{P}}(Z^{\mathbb{G}}_t \,|\, \mathcal{F}_t) = 1$ for every $t \in \mathbb{R}_+$.
Then there exists an equivalent probability measure $\bar{\mathbb{P}}$ such $\bar{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$ and the hypothesis (H) holds under $\bar{\mathbb{P}}$.

Proof. It suffices to show that (5.45) holds under $\bar{\mathbb{P}}$, where the probability measure $\bar{\mathbb{P}}$ is defined on $\mathcal{G}_t$ by $d\bar{\mathbb{P}}|_{\mathcal{G}_t} = Z^{\mathbb{G}}_t\, d\mathbb{P}|_{\mathcal{G}_t}$. We observe that, for all $0 \leq u \leq t$,

$$\begin{aligned}\bar{F}_{u,u} &= \mathbb{P}\left(Z^{\mathbb{G}}_u \mathbb{1}_{\{\tau\leq u\}} \,\middle|\, \mathcal{F}_u\right) = \mathbb{P}\left(\widehat{Z}_{\tau,u} \mathbb{1}_{\{\tau\leq u\}} \,\middle|\, \mathcal{F}_u\right) = (X_u)^{-1}\, \mathbb{P}\left(X_\tau \mathbb{1}_{\{\tau\leq u\}} \,\middle|\, \mathcal{F}_u\right) \\ &= (X_t)^{-1}\, \mathbb{P}\left(X_\tau \mathbb{1}_{\{\tau\leq u\}} \,\middle|\, \mathcal{F}_t\right) = \mathbb{P}\left(\widehat{Z}_{\tau,t} \mathbb{1}_{\{\tau\leq u\}} \,\middle|\, \mathcal{F}_t\right) = \mathbb{P}\left(Z^{\mathbb{G}}_t \mathbb{1}_{\{\tau\leq u\}} \,\middle|\, \mathcal{F}_t\right) = \bar{F}_{u,t}\end{aligned}$$

where the fourth equality is an immediate consequence of Lemma 5.7.4. □

An important conclusion from Proposition 5.7.3 is that if τ is a strictly positive random time such that the complete separability of the $(\mathbb{P},\mathbb{F})$-conditional distribution holds then, under technical conditions of Proposition 5.7.3, we have that $\mathcal{I}(\mathbb{P}) \neq \emptyset$. Therefore, part (ii) in Proposition 5.7.2 can be applied in these circumstances to the progressive enlargement $\mathbb{G}$.

5.7.2 Information Drift

For simplicity of presentation, we assume in this subsection that all $(\mathbb{P},\mathbb{F})$-local martingales are continuous, that is, assumption (**C**) is valid. Our aim is to apply the semimartingale decompositions developed in Section 5.6 to utility maximization and information theory associated with continuous time models of financial markets, as studied, in particular, by Ankirchner and Imkeller [2]. We will need the following definition borrowed from Ankirchner and Imkeller [2] (see Definition 1.1 in [2]). In what follows, the process X can be interpreted as the discounted price of a risky asset. The interested reader is referred to [2] for the compelling rationale for this definition in the context of maximization of the expected logarithmic utility from a portfolio's wealth when trading in X.

Definition 5.7.1. A filtration $\mathbb{F}$ is said to be a *finite utility filtration* for a process X whenever X is a $(\mathbb{P},\mathbb{F})$-semimartingale with the semimartingale decomposition of the form

$$X_t = U_t + \int_{(0,t]} \phi_u \, d\langle U, U\rangle_u \tag{5.50}$$

for some $(\mathbb{P},\mathbb{F})$-local martingale U and an $\mathbb{F}$-predictable process ϕ.

Remark 5.7.4. It was shown by Delbaen and Schachermayer [29] that for a locally bounded semimartingale X, the existence of an equivalent local martingale measure for X (or, equivalently, the property that X satisfies the No Free Lunch with Vanishing Risk (NFLVR) condition) implies that the decomposition of the process X must be of the form given in equation (5.50).

From now on, we work on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ with a filtration $\mathbb{F}$ satisfying the usual conditions and such that $\mathbb{F}$ is a finite utility filtration for a given process X so that decomposition (5.50) holds for some $(\mathbb{P},\mathbb{F})$-martingale U. The following definition comes from Imkeller [49] (see also Definition 1.2 in [2]).

Definition 5.7.2. Let $\mathbb{K}$ be any enlargement of the filtration $\mathbb{F}$. The $\mathbb{K}$-predictable process ψ such that the process

$$U_t - \int_{(0,t]} \psi_u \, d\langle U, U\rangle_u$$

is a $(\mathbb{P},\mathbb{K})$-local martingale is called the *information drift* of $\mathbb{K}$ with respect to $\mathbb{F}$.

Remark 5.7.5. It was shown in [2] (see Proposition 1.2 therein) that the difference of the maximal expected logarithmic utilities when trading in X is based on two different filtrations $\mathbb{F} \subset \mathbb{K}$ depends only on the information drift of $\mathbb{K}$ with respect to $\mathbb{F}$.

Under the standing assumption (**C**) all $(\mathbb{P},\mathbb{F})$-local martingales encountered in what follows are locally bounded, and thus locally square-integrable, so that we are in a position to apply the Kunita-Watanabe decomposition theorem (see, for instance, Protter [96]). We aim to show that the progressive enlargement of $\mathbb{F}$ with a pseudo-initial time τ is once again a finite utility filtration for X. We will also compute the information drift of the progressive enlargement $\mathbb{G}$ with respect to $\mathbb{F}$.

Proposition 5.7.4. *If $\mathbb{F}$ is a finite utility filtration for X and τ is a pseudo-initial time then the progressive enlargement $\mathbb{G}$ is a finite utility filtration for X. Furthermore, the information drift of $\mathbb{G}$ with respect to $\mathbb{F}$ is given by the following expression*

$$\psi_u = \mathbb{1}_{\{\tau \geq u\}}(G_u)^{-1}\eta_u + \mathbb{1}_{\{\tau < u\}}(m_{\tau,u})^{-1}\xi_{\tau,u} \tag{5.51}$$

where the $\mathbb{F}$-predictable processes $(\xi_{s,u})_{s\leq u}$ are defined by the Kunita-Watanabe decompositions

$$m_{s,t} = \int_{(s,t]} \xi_{s,u}\, dU_u + L^s_t \tag{5.52}$$

where L^s is a family of $(\mathbb{P},\mathbb{F})$-local martingales strongly orthogonal to U for all u.

Proof. By the standing assumption, the filtration $\mathbb{F}$ is a finite utility filtration for the process X. Therefore, the process X admits the $(\mathbb{P},\mathbb{F})$-semimartingale decomposition (see (5.50))

$$X_t = U_t + \int_{(0,t]} \phi_u\, d\langle U,U\rangle_u$$

where U is a $(\mathbb{P},\mathbb{F})$-local martingale and ϕ is an $\mathbb{F}$-predictable process. To establish the first assertion, it suffices to show that the process X is a $(\mathbb{P},\mathbb{G})$-semimartingale with the following decomposition

$$X_t = \bar{U}_t + \int_{(0,t]} \phi^*_u\, d\langle \bar{U},\bar{U}\rangle_u \tag{5.53}$$

where $\bar{U}$ is some $(\mathbb{P},\mathbb{G})$-martingale and ϕ^* is some $\mathbb{G}$-predictable process. Using Corollary 5.6.5 and the assumption that all $(\mathbb{P},\mathbb{F})$-local martingales are continuous (so that $\widetilde{C} = \langle U,M\rangle$), we deduce that the process $\bar{U}$, which is given by the expression

$$\bar{U}_t = U_t - \int_{(0,t]} \mathbb{1}_{\{\tau\geq u\}}(G_u)^{-1}\, d\langle U,M\rangle_u - \int_{(t\wedge s,t]} (m_{s,u-})^{-1}\, d\langle U, m_{s,\cdot}\rangle_u \big|_{s=\tau}, \tag{5.54}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale. Recall that we denote by M the $(\mathbb{P},\mathbb{F})$-local martingale appearing in the Doob-Meyer decomposition of G. An application of the Kunita-Watanabe decomposition theorem to M and U, after suitable localization if required, gives

$$M_t = \int_{(0,t]} \eta_u\, dU_u + \widehat{L}_t \tag{5.55}$$

where η is some $\mathbb{F}$-predictable process and a square-integrable $(\mathbb{P},\mathbb{F})$-martingale $\widehat{L}$ is strongly orthogonal to U. It thus follows immediately from (5.55) that

$$d\langle U,M\rangle_u = \eta_u\, d\langle U,U\rangle_u.$$

In the next step, we focus on the $(\mathbb{P},\mathbb{F})$-martingale $(m_{s,t})_{t\geq s}$, for any fixed $s\in\mathbb{R}_+$. Using once again the Kunita-Watanabe decomposition theorem, we deduce that (5.52) holds for all $s\leq t$ where $(\xi_{s,u})_{u\geq s}$ is a family of $\mathbb{F}$-predictable processes parametrized by s and L^s a family of $(\mathbb{P},\mathbb{F})$-local martingales strongly orthogonal to U for every s. Consequently, by combining (5.52), (5.54) and (5.55), we arrive at the following equalities

$$\begin{aligned}\bar{U}_t &= U_t - \int_{(0,t]} \mathbb{1}_{\{\tau\geq u\}}(G_u)^{-1}\eta_u\, d\langle U,U\rangle_u + \int_{(0,t]} \mathbb{1}_{\{\tau<u\}}(m_{\tau,u})^{-1}\xi_{\tau,u}\, d\langle U,U\rangle_u \\ &= U_t - \int_{(0,t]} \Big(\mathbb{1}_{\{\tau\geq u\}}(G_u)^{-1}\eta_u + \mathbb{1}_{\{\tau<u\}}(m_{\tau,u})^{-1}\xi_{\tau,u}\Big)\, d\langle U,U\rangle_u.\end{aligned}$$

It is clear that the equality $\langle U,U\rangle = \langle \bar{U},\bar{U}\rangle$ holds. Therefore, the canonical $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of U reads

$$U_t = \bar{U}_t + \int_{(0,t]} \psi_u\, d\langle \bar{U},\bar{U}\rangle_u \tag{5.56}$$

where the $\mathbb{G}$-predictable process ψ is given by the following expression

$$\psi_u = \mathbb{1}_{\{\tau\geq u\}}(G_u)^{-1}\eta_u + \mathbb{1}_{\{\tau<u\}}(m_{\tau,u})^{-1}\xi_{\tau,u}.$$

It is now easy to see that the $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of X has indeed the desired form (5.53) with $\phi^* = \phi + \psi$. To derive equality (5.51), it is enough to employ Definition 5.7.2. The asserted formula follows directly from representation (5.56) and the fact that $\langle \bar{U},\bar{U}\rangle = \langle U,U\rangle$. □

Chapter 6

Stability of Random Times under Min and Max

This chapter was developed during my visit to Université d'Evry in 2011 and is based on the working paper with the same title by Jeanblanc, Li and Song [58]. I would like to point out that, although this result is not included here, it was shown by Song and Aksamit at the time that the minimum and maximum of two honest time is again an honest time.

6.1 Introduction

In this chapter, we work on the usual filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Suppose T_1 and T_2 are $\mathbb{F}$-stopping time, then it is well known that $T_1 \wedge T_2$ and $T_1 \vee T_2$ are $\mathbb{F}$-stopping times. Inspired by this observation, working under the context of progressive enlargement of filtrations, we examine in this chapter the stability of the hypothesis (H), the hypothesis (H'), the class of honest times and pseudo stopping times under minimum and maximum. More specifically, given two finite random times τ_i for $i = 1, 2$ we address the following question: if the hypothesis (H) or (H') is satisfied between $\mathbb{F}$ and the progressive enlargements of $\mathbb{F}$ with τ_i for $i = 1, 2$, then under what conditions are the hypothesis (H) or (H') satisfied between $\mathbb{F}$ and the progressive enlargement of $\mathbb{F}$ with either the random time $\tau_1 \wedge \tau_2$ or $\tau_1 \vee \tau_2$.

Perhaps hidden in the investigation of the above mentioned question is the relationship between the Cox construction, the hypothesis (H) and pseudo-stopping times. It is well known (see, for example, Bielecki et al. [14]) that if a random time is constructed through the Cox construction then it satisfies the hypothesis (H). It was also shown in Nikeghbali and Yor [92] that if a random time τ satisfies the hypothesis (H) then it is a pseudo-stopping time (see Definition 6.4.1). That is, the following implications hold:

$$\text{Cox construction} \implies \text{Hypothesis } (H) \implies \text{Pseudo-stopping time.}$$

Parts of this section are aimed to provide some partial converses to the above implications. First, Lemma 6.2.6 (an extension of Norros lemma; see Gapeev and Jeanblanc [43] and the references therein) demonstrates that if a random time τ satisfies the hypothesis (H) and the dual optional projection of $H = \mathbb{1}_{[\![\tau,\infty[\![}$ is continuous, then τ can be obtained through the Cox construction. Second, we show in in Lemma 6.4.3 that, under assumption $(\mathbf{C})$, if τ is a pseudo-stopping time satisfying the hypothesis (HP) then τ satisfies the hypothesis (H).

For the reader's convenience, we recall the definitions of hypotheses (H) and (H').

Definition 6.1.1. We say that the *hypothesis* (H) is satisfied by the filtrations $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$ if any $(\mathbb{P},\mathbb{F})$-martingale is also a $(\mathbb{P},\mathbb{K})$-martingale. We sometimes write $\mathbb{F} \hookrightarrow \mathbb{K}$ or say that $\mathbb{F}$ is *immersed* in $\mathbb{K}$.

Definition 6.1.2. We say that the *hypothesis* (H') is satisfied by the filtrations $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$ if any $(\mathbb{P},\mathbb{F})$-semimartingale is also a $(\mathbb{P},\mathbb{K})$-semimartingale.

Let us introduce the notation used throughout the section. Given a filtration $\mathbb{F}$ and two finite random times τ_1 and τ_2, we set $H^i = \mathbb{1}_{[\![\tau_i,\infty[\![}$ and we denote the progressive enlargements of $\mathbb{F}$ by $\mathbb{F}^{\tau_i} = (\mathcal{F}_t^{\tau_i})_{t\geq 0}$ and $\mathbb{F}^{\tau_1,\tau_2} = (\mathcal{F}_t^{\tau_1,\tau_2})_{t\geq 0}$ for $i = 1, 2$. To be more specific, we set

$$\mathcal{F}_t^{\tau_i} = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau_i \wedge s),$$
$$\mathcal{F}_t^{\tau_1,\tau_2} = \bigcap_{s>t} \mathcal{F}_s^{\tau_1} \vee \sigma(\tau_2 \wedge s) = \bigcap_{s>t} \mathcal{F}_s^{\tau_2} \vee \sigma(\tau_1 \wedge s).$$

For an arbitrary filtration $\mathbb{F}$, the $\mathbb{F}$-optional projection of the process $1-H^i$ is given by $\mathbb{P}\left(\tau_i > t \,\middle|\, \mathcal{F}_t\right) = G_t^{i,\mathbb{F}}$ and dual $\mathbb{F}$-optional projection of H^i is denoted by $\bar{A}^{i,\mathbb{F}}$. If there is no confusion with the random time and the filtration, we shall often omit the superscript ${}^{i,\mathbb{F}}$.

Definition 6.1.3. For a random time τ, by the $\mathbb{F}$-*conditional distribution of* τ is the random field

$$F_{u,t} = \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t), \quad \forall\, u, t \in \bar{\mathbb{R}}_+,$$

and by the $\mathbb{F}$-*conditional survival distribution of* τ is the random field

$$G_{u,t} = \mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = 1 - F_{u,t}, \quad \forall\, u, t \in \bar{\mathbb{R}}_+.$$

Definition 6.1.4. A random time τ is said to satisfy the *hypothesis* (HP) whenever for all $0 \leq u \leq s \leq t$

$$F_{u,s}F_{s,t} = F_{s,s}F_{u,t}. \tag{6.1}$$

We also recall the following result (see Brémaud and Yor [20]).

Lemma 6.1.1. *Assume that $\mathbb{F} \subset \mathbb{K}$, that is, a filtration $\mathbb{F}$ is a sub-filtration of $\mathbb{K}$. Then $\mathbb{F} \hookrightarrow \mathbb{K}$ if and only if any of the following equivalent conditions holds:*
(i) for any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{K}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$, that is, for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and any bounded, $\mathcal{K}_t$-measurable random variable η we have

$$\mathbb{E}_{\mathbb{P}}\left(\xi\eta \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\xi \,\middle|\, \mathcal{F}_t\right) \mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_t\right), \tag{6.2}$$

(ii) for any $t \in \mathbb{R}_+$ and any $u \geq t$, the σ-fields $\mathcal{F}_u$ and $\mathcal{K}_t$ are conditionally independent given $\mathcal{F}_t$,
(iii) for any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ

$$\mathbb{E}_{\mathbb{P}}\left(\xi \,\middle|\, \mathcal{K}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\xi \,\middle|\, \mathcal{F}_t\right), \tag{6.3}$$

(iv) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{K}_t$-measurable random variable η

$$\mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \mathcal{F}_\infty\right), \tag{6.4}$$

(v) In particular, if $\mathbb{K}$ is the progressive enlargement of $\mathbb{F}$ with τ, then for any $t \in \mathbb{R}_+$

$$\mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right) = \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_\infty\right). \tag{6.5}$$

6.2 Hypothesis (H) under Min and Max

In the following, we work under the setting of progressive enlargement. The basic assumption is that the filtration $\mathbb{F}$ is immersed in both $\mathbb{F}^{\tau_1}$ and $\mathbb{F}^{\tau_2}$ and the aim is to find conditions so that $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1\wedge\tau_2}$ or $\mathbb{F}^{\tau_1\vee\tau_2}$.

6.2.1 Preliminary Results

We first present several conditions under which $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1\wedge\tau_2}$. These conditions appear to be difficult to check and thus their applications are limited, however.

Lemma 6.2.1. *Assume that the filtration $\mathbb{F}$ is immersed in both $\mathbb{F}^{\tau_1}$ and $\mathbb{F}^{\tau_2}$. Then $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1\wedge\tau_2}$ if and only if $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1\vee\tau_2}$.*

Proof. It is clear that

$$\mathbb{P}\left(\tau_1\vee\tau_2>t\,\middle|\,\mathcal{F}_t\right)=\mathbb{P}\left(\tau_1>t\,\middle|\,\mathcal{F}_t\right)+\mathbb{P}\left(\tau_2>t\,\middle|\,\mathcal{F}_t\right)-\mathbb{P}\left(\tau_1\wedge\tau_2>t\,\middle|\,\mathcal{F}_t\right),$$

and thus the claim follows from part (v) of Lemma 6.1.1. □

In view of Lemma 6.2.1, it suffices to work with $\tau_1\wedge\tau_2$.

Lemma 6.2.2. *Let τ_1 be random time such that $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1}$. If τ_2 is an $\mathbb{F}^{\tau_1}$-stopping time then $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1,\tau_2}$.*

Proof. The claim is obvious, since if τ_2 is an $\mathbb{F}^{\tau_1}$-stopping time then $\mathbb{F}^{\tau_1}=\mathbb{F}^{\tau_1,\tau_2}$. □

Lemma 6.2.3. *If $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1,\tau_2}$ then $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1}$, $\mathbb{F}^{\tau_2}$, $\mathbb{F}^{\tau_1\vee\tau_2}$ and $\mathbb{F}^{\tau_1\wedge\tau_2}$.*

Proof. Since τ_1 and τ_2 are $\mathbb{F}^{\tau_1,\tau_2}$-stopping times, $\tau_1\wedge\tau_2$ is a $\mathbb{F}^{\tau_1,\tau_2}$-stopping time and the set $\{\tau_1\wedge\tau_2>t\}$ is $\mathcal{F}_t^{\tau_1,\tau_2}$-measurable. Therefore, the result follows immediately by the assumption that $\mathbb{F}\hookrightarrow\mathbb{F}^{\tau_1,\tau_2}$. □

We give some necessary and/or sufficient conditions to have $\mathbb{F}$ immersed in $\mathbb{F}^{\tau_1,\tau_2}$.

Lemma 6.2.4. *The filtration $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1,\tau_2}$ if and only if for all $t\in\mathbb{R}_+$, $c\le t$ and $k\le t$,*

$$\mathbb{P}\left(\tau_1\le c\,,\tau_2\le k\,\middle|\,\mathcal{F}_t\right)=\mathbb{P}\left(\tau_1\le c,\tau_2\le k\,\middle|\,\mathcal{F}_\infty\right)\tag{6.6}$$

Proof. The 'only if' part is clear in view part (iv) of Lemma 6.1.1. Therefore, to see that the immersion holds between $\mathbb{F}$ and $\mathbb{F}^{\tau_1,\tau_2}$, it is now sufficient to notice that for any bounded, $\mathcal{F}_t$-measurable random variable x_t

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(x_tg(\tau_1\wedge t)h(\tau_2\wedge t)\,\middle|\,\mathcal{F}_t\right)&=\int_{[0,\infty]^2}x_tg(u\wedge t)h(s\wedge t)\,d\mathbb{P}\left(\tau_1\le u\,,\tau_2\le s\,\middle|\,\mathcal{F}_t\right)\\&=\int_{[0,\infty]^2}x_tg(u\wedge t)h(s\wedge t)\,d\mathbb{P}\left(\tau_1\le u\,,\tau_2\le s\,\middle|\,\mathcal{F}_\infty\right)\\&=\mathbb{E}_{\mathbb{P}}\left(x_tg(\tau_1\wedge t)h(\tau_2\wedge t)\,\middle|\,\mathcal{F}_\infty\right)\end{aligned}$$

where the second equality holds by (6.6). □

Proposition 6.2.1. *Let τ_1 and τ_2 be random times such that:*
(i) the filtration $\mathbb{F}$ is immersed in both $\mathbb{F}^{\tau_1}$ and $\mathbb{F}^{\tau_2}$,
(ii) the random times τ_1 and τ_2 are conditionally independent given $\mathcal{F}_\infty$.
Then the filtration $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1,\tau_2}$.

Proof. By condition (ii), we have, for all $c\le t$ and $k\le t$,

$$\mathbb{P}\left(\tau_1>c,\tau_2>k\,\middle|\,\mathcal{F}_t\right)=\mathbb{P}\left(\mathbb{P}\left(\tau_1>c,\tau_2>k\,\middle|\,\mathcal{F}_\infty\right)\,\middle|\,\mathcal{F}_t\right)=\mathbb{E}_{\mathbb{P}}\left(\mathbb{P}\left(\tau_1>c\,\middle|\,\mathcal{F}_\infty\right)\mathbb{P}\left(\tau_2>k\,\middle|\,\mathcal{F}_\infty\right)\,\middle|\,\mathcal{F}_t\right).$$

Since both τ_1 and τ_2 satisfy the hypothesis (H), we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(\mathbb{P}\left(\tau_1>c\,\middle|\,\mathcal{F}_\infty\right)\mathbb{P}\left(\tau_2>k\,\middle|\,\mathcal{F}_\infty\right)\,\middle|\,\mathcal{F}_t\right)&=\mathbb{P}\left(\tau_1>c\,\middle|\,\mathcal{F}_t\right)\mathbb{P}\left(\tau_2>k\,\middle|\,\mathcal{F}_t\right)\\&=\mathbb{P}\left(\tau_1>c\,\middle|\,\mathcal{F}_\infty\right)\mathbb{P}\left(\tau_2>k\,\middle|\,\mathcal{F}_\infty\right)\\&=\mathbb{P}\left(\tau_1>c,\tau_2>k\,\middle|\,\mathcal{F}_\infty\right)\end{aligned}$$

where the last equality holds again by conditional independence of $\mathbb{F}^{\tau_1}$ and $\mathbb{F}^{\tau_2}$ given $\mathcal{F}_\infty$. □

6.2.2 Extended Canonical Construction

Let us recall the *extended canonical construction* (see Li and Rutkowski [74]), which assumes that we are given the following probabilistic data: (i) a (non-adapted) non-decreasing, càdlàg stochastic process $(D_t)_{t\geq 0}$, (ii) a random variable U uniformly distributed on $(0,1)$ and independent of $\mathcal{F}_\infty$.

A random time τ can then be constructed by setting

$$\tau := \inf\{t : D_t \geq U\} = C_{U-} \tag{6.7}$$

with the process C is the generalized right inverse of D. If the probability space is not rich enough to support the existence of such U in (ii), one can always extend the original space to ensure such U exists. Obviously, any random time τ given on Ω can be trivially represented in this way without extending the original space, since it is enough to select $D_t = \mathbb{1}_{\{\tau\leq t\}}$ in (6.7).

If we impose the rule that in (i), the non-decreasing random function D is a $\mathbb{F}$-adapted process, then we recover the canonical construction and the random time constructed in this way necessarily satisfis the hypothesis (H).

Let us first state the Norros lemma in the form given by Proposition 3.1 of Gapeev and Jeanblanc [43]. In the following, Λ denotes the $\mathbb{F}^\tau$-predictable compensator of $H = \mathbb{1}_{[\![\tau,\infty[\![}$.

Proposition 6.2.2. *Let the process $G_t = \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right)$ be continuous and such that $G_0 = 1$. Then:*
(i) the random variable Λ_τ has the standard exponential law (with parameter 1);
(ii) if $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$ then Λ_τ is independent of $\mathcal{F}_\infty$.

In the following, we provide an extension to the Norros lemma stated above. Our method is based on stochastic calculus and the time change formula. For an arbitrary random time τ, we set $\widehat{D}_t := \mathbb{P}\left(\tau \leq t \,\middle|\, \mathcal{F}_\infty\right)$ and we define $\widehat{C}$ as the generalized right inverse of $\widehat{D}$.

Proposition 6.2.3. *The random variable $\widehat{D}_\tau$ is independent from $\mathcal{F}_\infty$ if and only if the random function $h(x) := \sup\{t : \widehat{D}_{\widehat{C}_t} \leq x\}$ is deterministic.*

Proof. We have, for every $x \geq 0$,

$$\begin{aligned}
\mathbb{P}(\widehat{D}_\tau \leq x \,|\, \mathcal{F}_\infty) &= \int_{[0,\infty]} \mathbb{1}_{\{\widehat{D}_u \leq x\}}\, d\widehat{D}_u \\
&= \int_{[0,\infty]} \mathbb{1}_{\{\widehat{D}_{\widehat{C}_u} \leq x\}} \mathbb{1}_{\{\widehat{C}_u < \infty\}}\, du \\
&= \int_{[0,\infty]} \mathbb{1}_{\{\widehat{D}_{\widehat{C}_u} \leq x\}} \mathbb{1}_{\{u<1\}}\, du \\
&= \sup\{t : \widehat{D}_{\widehat{C}_t} \leq x\} \wedge 1.
\end{aligned}$$

Hence if h is deterministic then $\mathbb{P}(\widehat{D}_\tau \leq x \,|\, \mathcal{F}_\infty) = \mathbb{P}(\widehat{D}_\tau \leq x)$ for all $x \geq 0$. □

Corollary 6.2.1. *If process $(\widehat{D}_t)_{t\geq 0}$ is continuous then the random variable $\widehat{D}_\tau$ is uniformly distributed and independent from $\mathcal{F}_\infty$.*

Proof. If the process $\widehat{D}$ is continuous then

$$\mathbb{P}(\widehat{D}_\tau \leq x \,|\, \mathcal{F}_\infty) = \sup\{t : t \leq x\} \wedge 1 = x \wedge 1$$

since $\widehat{C}$ is the right inverse of $\widehat{D}$ so that $\widehat{D}_{\widehat{C}_t} = t$ for all $t \geq 0$. □

Corollary 6.2.1 be seen as an extension of the result proven in [43]. Indeed, if τ satisfies the hypothesis (H), then $\widehat{D}_t = \mathbb{P}\left(\tau \leq t \,\middle|\, \mathcal{F}_\infty\right) = \mathbb{P}\left(\tau \leq t \,\middle|\, \mathcal{F}_t\right) = F_t$ and thus we recover the result from [43]. Let us now give an example where $\widehat{D}_\tau$ is independent from $\mathcal{F}_\infty$, but fails to be continuous.

Example 6.2.1. Consider the probability space $\Omega = \{\omega_1, \omega_2\}$ with a filtration $\mathbb{F}$ such that the strictly increasing process $\widehat{D}$, which is given by

$$\widehat{D}_t(\omega_1) = 1 - e^{-(t+\mathbb{1}_{\{t\geq 1\}})} \quad \text{and} \quad \widehat{D}_t(\omega_2) = 1 - e^{-(t^2+\mathbb{1}_{\{t\geq 1\}})},$$

is $\mathbb{F}$-adapted. One can then construct a random time τ by setting $\tau := \inf\{t : \widehat{D}_t > U\}$ where U is uniformly distributed and independent of $\mathcal{F}_\infty$. In this situation, the filtration $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$ and the dual optional projection $\bar{A}$ is equal to $\widehat{D}$. One can easily check that $h(x) = \sup\{t : \widehat{D}_{\widehat{C}_t} \leq x\}$ is deterministic.

The canonical construction is frequently referred to as the Cox construction in the literature. This terminology motivates the following definition.

Definition 6.2.1. A random time τ is called an $\mathbb{F}$-*Cox-time* if there exists a càdlàg $\mathbb{F}$-adapted non-decreasing process X and a random variable U independent of $\mathcal{F}_\infty$ such that

$$\tau = \inf\{t : X_t \geq U\} = C_{U-}$$

where C and X are the generalized right-inverses of one another.

Proposition 6.2.4. *If the random times τ_1 and τ_2 are $\mathbb{F}$-Cox-times then $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1,\tau_2}$.*

Proof. The fact that the random times τ_i are $\mathbb{F}$-Cox-times implies the existence of $\mathbb{F}$-adapted and non-decreasing processes X^i such that

$$\tau_1 = \inf\{t : X_t^1 \geq U^1\} \quad \text{and} \quad \tau_2 = \inf\{t : X_t^2 \geq U^2\},$$

where the random variables U^i for $i = 1, 2$ are both independent of $\mathcal{F}_\infty$. Therefore,

$$\mathbb{P}(\tau_1 \leq c,\, \tau_2 \leq k \,|\, \mathcal{F}_\infty) = \mathbb{P}(U^1 \leq X_c^1,\, U^2 \leq X_k^2 \,|\, \mathcal{F}_\infty),$$
$$\mathbb{P}(\tau_1 \leq c,\, \tau_2 \leq k \,|\, \mathcal{F}_t) = \mathbb{P}(U^1 \leq X_c^1,\, U^2 \leq X_k^2 \,|\, \mathcal{F}_t).$$

Then, as U^i for $i = 1, 2$ are independent from $\mathcal{F}_\infty$ (so from $\mathcal{F}_t$ as well), we have, for all $c, k \leq t$,

$$\mathbb{P}(\tau_1 \leq c,\, \tau_2 \leq k \,|\, \mathcal{F}_\infty) = \psi(X_c^1, X_k^2) = \mathbb{P}(\tau_1 \leq c,\, \tau_2 \leq k \,|\, \mathcal{F}_t)$$

where $\psi(x, y) = \mathbb{P}(U^1 \leq x, U^2 \leq y)$. □

We know from Bielecki et al. [14] that a Cox-time satisfies the hypothesis (H). Therefore, if one could show the converse then, by Proposition 6.2.4, the filtration $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1,\tau_2}$.

Let us first make the following two observations:

- If τ is a Cox-time and X is continuous then X is also the left inverse of C and we can recover U from (6.7) by $U = X \circ C_{U-} = X_\tau$ and $\tau = \inf\{t : X_t \geq X_\tau\}$.
- If X is a strictly increasing process then a random time (not necessarily a Cox-time) τ can always be represented as $\tau = \inf\{t : X_t \geq X_\tau\}$ and $C_{X_\tau} = \tau$ holds.

As already mentioned, we denote the $\mathbb{F}$-dual optional projection of $H := \mathbb{1}_{[\![\tau,\infty[\![}$ by $\bar{A}$. The next lemma shows that, in view of above observations, it is in some sense natural to study the random variable $\bar{A}_\tau$.

Lemma 6.2.5. *Let $\tau = \inf\{t : X_t \geq X_\tau\}$ for some $\mathbb{F}$-adapted non-decreasing càdlàg process X with X_τ independent from $\mathcal{F}_\infty$. Then $\bar{A}_\tau$ is independent from $\mathcal{F}_\infty$.*

Proof. One can easily check that $F_t = \mathbb{P}\left(X_t \geq X_\tau \,\middle|\, \mathcal{F}_\infty\right)$ since the random time τ satisfies the hypothesis (H) and X_τ is independent from $\mathcal{F}_\infty$. On the other hand,

$$\mathbb{P}\left(X_t \geq X_\tau \,\middle|\, \mathcal{F}_\infty\right) = \mathbb{P}(X_\tau \leq u)|_{u=X_t} = \Psi(X_t)$$

where Ψ is the distribution function of X_τ. The hypothesis (H) tells us that $F_t = \bar{A}_t$, which implies that for every $t \in \mathbb{R}_+$,

$$\bar{A}_t = \mathbb{P}(X_\tau \leq u)|_{u=X_t} = \Psi(X_t).$$

The processes $(F(X_t))_{t\geq 0}$ and $(\bar{A}_t)_{t\geq 0}$ are indistinguishable, since both are càdlàg processes. This implies that $\bar{A}_\tau$ must be independent from $\mathcal{F}_\infty$, as for any bounded, Borel function h, we have

$$\mathbb{E}_{\mathbb{P}}\left(h(\bar{A}_\tau) \,\middle|\, \mathcal{F}_\infty\right) = \mathbb{E}_{\mathbb{P}}\left(h(\Psi(X_\tau)) \,\middle|\, \mathcal{F}_\infty\right) = \mathbb{E}_{\mathbb{P}}\left(h(\Psi(X_\tau))\right) = \mathbb{E}_{\mathbb{P}}\left(h(\bar{A}_\tau)\right)$$

where the third equality holds by the assumed independence of X_τ from $\mathcal{F}_\infty$. □

Lemma 6.2.6. *If the random time τ satisfies hypothesis (H) and avoids all $\mathbb{F}$-stopping times then $A_\tau = \bar{A}_\tau = F_\tau$ is uniformly distributed and independent of $\mathcal{F}_\infty$.*

Proof. If τ avoids all $\mathbb{F}$-stopping times then the process F is continuous. Hence in Proposition 6.2.3 the process $\widehat{C}$ is in fact the right inverse of F, which implies

$$\mathbb{P}\left(F_\tau \leq k \,\middle|\, \mathcal{F}_\infty\right) = \sup\{t : t \leq k\} \wedge 1 = k \wedge 1.$$

Therefore, $\bar{A}_\tau$ is uniformly distributed on $(0,1)$ and it is independent of $\mathcal{F}_\infty$. □

Lemma 6.2.7. *If τ is a random time that satisfies hypothesis (H) and $\bar{A}_\tau$ is uniformly distributed on $(0,1)$ and independent from $\mathcal{F}_\infty$ then τ is a Cox-time.*

Proof. Given a random time τ, we first define another random time τ^* by setting

$$\tau^* := \inf\{t : \bar{A}_t \geq \bar{A}_\tau\} \tag{6.8}$$

where $\bar{A}$ is the dual $\mathbb{F}$-optional projection of $\mathbb{1}_{[\![\tau,\infty[\![}$. It is easy to see that $\tau^* \leq \tau$. It is sufficient to show that $\{\tau^* < \tau\}$ is of measure zero.

For this purpose, we note that for all $t \in \mathbb{R}_+$ we have $\{\tau \leq t\} \subset \{\tau^* \leq t\}$. Since $\bar{A}_\tau$ is uniformly distributed, the $\mathbb{F}$-conditional distributions of τ^* and τ are the same. This implies that, for all $t \in \mathbb{R}_+$, the set $\{\tau \leq t\}$ is almost surely equal to $\{\tau^* \leq t\}$ or, equivalently, the set $\{\tau^* > t\}$ is almost surely equal to $\{\tau > t\}$. Therefore, almost sure we have

$$\mathbb{1}_{\{\tau^* < \tau\}} \leq \sum_{q\in\mathbb{Q}} \mathbb{1}_{\{\tau > q\}}\mathbb{1}_{\{\tau^* < q\}} = \sum_{q\in\mathbb{Q}} \mathbb{1}_{\{\tau^* > q\}}\mathbb{1}_{\{\tau^* < q\}} = 0.$$

This implies the set $\{\tau^* < \tau\}$ has measure zero. □

Corollary 6.2.2. *If τ is a random time such that the filtration $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1}$ and it avoids all $\mathbb{F}$-stopping times then τ is a Cox-time.*

Proof. By Lemma 6.2.6, $\bar{A}_\tau$ is uniformly distributed and independent from $\mathcal{F}_\infty$. The result follows from Lemma 6.2.7. □

6.3 Hypothesis (H') under Min and Max

In this section, we investigate the invariance of hypothesis (H') under the operations $\wedge$ and $\vee$. To simplify language, we say that a random time τ satisfies the hypothesis (H') when the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^\tau$. We start the investigation with the following result based on the deep results of Stricker [106].

6.3.1 Preliminary Results

The following lemma is an immediate consequence of Definition 6.1.2.

Lemma 6.3.1. *Assume that the filtrations $\mathbb{F}$, $\mathbb{G}$ and $\mathbb{K}$ are such that $\mathbb{F} \subset \mathbb{G} \subset \mathbb{K}$. If the hypothesis (H') is satisfied by $\mathbb{F}$ and $\mathbb{K}$ then the hypothesis (H') also holds between $\mathbb{F}$ and $\mathbb{G}$.*

Proposition 6.3.1. *If τ satisfies the hypothesis (H'), then for any $\mathbb{F}^{\tau}$-stopping time ϑ, the random time $\tau \wedge \vartheta$ and $\tau \vee \vartheta$ also satisfies the hypothesis (H').*

Proof. We shall give the proof for $\tau \wedge \vartheta$ only, since the proof for $\tau \vee \vartheta$ is similar. We note that $\tau \wedge \vartheta$ is an $\mathbb{F}^{\tau}$-stopping times. This implies $\mathbb{F}^{\tau \wedge \theta} \subset \mathbb{F}^{\tau}$ and the result now follows from Lemma 6.3.1. □

6.3.2 General Case

Let us consider any two random times τ_1 and τ_2 defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$ satisfying the usual conditions.

The aim in the following is to show that if τ_1 and τ_2 satisfy the hypothesis (H') then the hypothesis (H') is also satisfied by the random times $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$. We shall only give an outline of this approach for $\tau_1 \wedge \tau_2$, since the method is similar for $\tau_1 \vee \tau_2$.

Remark 6.3.1. Before proceeding, let us point that the technique developed in the following is inspired by the following observation: if V_t is $\mathcal{F}_t^{\tau_1 \wedge \tau_2}$-measurable then there exists a $\sigma(\tau_1 \wedge \tau_2) \vee \mathcal{F}_t$-measurable map $V(u,t)$ and $\widehat{V}_t \in \mathcal{F}_t$ so that

$$
\begin{aligned}
V_t &= \widehat{V}_t \mathbb{1}_{\{\tau_1 \wedge \tau_2 > t\}} + \lim_{s \downarrow t} V(\tau_1 \wedge \tau_2, s) \mathbb{1}_{\{\tau_1 \wedge \tau_2 \le t\}} \\
&= \left(\widehat{V}_t \mathbb{1}_{\{\tau_1 > t\}} + \lim_{s \downarrow t} V(\tau_1, s) \mathbb{1}_{\{\tau_1 \le t\}} \right) \mathbb{1}_{\{\tau_1 \le \tau_2\}} + \left(\widehat{V}_t \mathbb{1}_{\{\tau_2 > t\}} + \lim_{s \downarrow t} V(\tau_2, s) \mathbb{1}_{\{\tau_2 \le t\}} \right) \mathbb{1}_{\{\tau_1 > \tau_2\}} \\
&= V_t^{\tau_1} \mathbb{1}_{\{\tau_1 \le \tau_2\}} + V_t^{\tau_2} \mathbb{1}_{\{\tau_1 > \tau_2\}}
\end{aligned}
$$

where for $i = 1, 2$, the process V^{τ_i} is $\mathbb{F}_t^{\tau_i}$-adapted.

Intuitively, Remark 6.3.1 appears to be telling us that the progressive enlargement of $\mathbb{F}$ with $\tau_1 \wedge \tau_2$ can be viewed as $\mathbb{F}^{\tau_1}$ on $\{\tau_1 \le \tau_2\}$ and $\mathbb{F}^{\tau_2}$ on $\{\tau_1 > \tau_2\}$. This motivates the following construction.

Suppose that $\mathcal{D} = \{D_1, \dots, D_k\}$ is a family of $\mathcal{F}$-measurable sets which covers Ω and $\mathbb{F}^i$ for $i = 1, \dots, k$ is a family of filtrations satisfying the usual conditions. We define, for every $t \ge 0$, the following auxiliary family of sets

$$
\widehat{\mathcal{F}}_t := \{ A \in \mathcal{G} \,|\, \forall i, \, \exists A_t^i \in \mathcal{F}_t^i \text{ such that } A \cap D_i = A_t^i \cap D_i \}.
$$

Our goal is now to show that $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t \ge 0}$ is a right-continuous filtration and the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}$ if it is satisfied between $\mathbb{F}$ and all $\mathbb{F}^i$. Without loss of generality, we may and do assume that the sets from $\mathcal{D}$ are pairwise disjoint. Otherwise, one can always define for $i = 1, \dots k$, the disjoint sets $D_i^* := D_i \cap (\cup_{k<i} D_k)^c$ and $\mathcal{D}^* = \{D_1^*, \dots, D_k^*\}$ is a partition of Ω. One can then define the family of sets

$$
\widehat{\mathcal{F}}_t^* := \{ A \in \mathcal{G} \,|\, \forall i, \, \exists A_t^i \in \mathcal{F}_t^i \text{ such that } A \cap D_i^* = A_t^i \cap D_i^* \}
$$

and it is easy to see that $\widehat{\mathcal{F}}_t$ is contained in $\widehat{\mathcal{F}}_t^*$. If we can show that the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}^*$, then by Lemma 6.3.1, the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}$.

The above construction and the subsequent semimartingale decomposition formula derived in Lemma 6.3.6 can be viewed as an extension of the idea used in the study of initial enlargement by

Meyer [84] and Yor [109]. In fact, if we assume that for all $i = 1, \dots, k$, the filtrations $\mathbb{F}^i = \mathbb{F}$ then we can recover from Lemma 6.3.6 the semimartingale decomposition result given in [84] and [109].

We first show, using Lemma 6.3.2 and Lemma 6.3.4, that $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{t\geq 0}$ is a right-continuous filtration.

Lemma 6.3.2. *For every $t \geq 0$, the family of sets $\widehat{\mathcal{F}}_t$ is an σ-algebra.*

Proof. It is obvious that the empty set and Ω belong to $\widehat{\mathcal{F}}_t$. To see that $\widehat{\mathcal{F}}_t$ is closed under countable unions and intersections, we simply apply the distribution law. We will now check that it is closed under complements. Suppose $A \in \widehat{\mathcal{F}}_t$ then there exists by definition a set $A^i_t \in \mathcal{F}^i_t$ such that

$$A \cap D_i = A^i_t \cap D_i.$$

Take the complement of both sides in the above and by De Morgan's law

$$A^c \cup D_i^c = (A^i_t)^c \cup D_i^c.$$

Next, by taking intersection with the set D^i on both sides of the above, we arrive at

$$A^c \cap D_i = (A^i_t)^c \cap D_i$$

and this concludes the proof. □

Proposition 6.3.2. *The inclusion $D_i \subseteq \{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right) > 0\}$ holds $\mathbb{P}$-a.s.*

Proof. We prove the inclusion $D_i \subset \{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right) > 0\}$. Let us consider

$$\mathbb{P}(A^i) = \int_{A^i} d\mathbb{P} = \int_{A^i} \mathbb{P}\left(D_i^c \,\middle|\, \mathcal{F}^i_t\right) d\mathbb{P} = \mathbb{P}(D_i^c \cap A^i)$$

which implies that the inclusion $\{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right) = 0\} \subseteq D_i^c$ holds, $\mathbb{P}$-a.s. This implies that the inclusion $D_i \subseteq \{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right) > 0\}$ holds, $\mathbb{P}$-a.s. □

Lemma 6.3.3. *Let η be any integrable random variable. Then, for every $i = 1, \dots, k$,*

$$\mathbb{E}_{\mathbb{P}}\left(\eta \mathbb{1}_{D_i} \,\middle|\, \widehat{\mathcal{F}}_t\right) = \mathbb{1}_{D_i} \frac{\mathbb{E}_{\mathbb{P}}\left(\eta \mathbb{1}_{D_i} \,\middle|\, \mathcal{F}^i_t\right)}{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right)} \tag{6.9}$$

for every $t \geq 0$.

Proof. Let $\widehat{F}_t \in \widehat{\mathcal{F}}_t$, then there exists $F^i_t \in \mathcal{F}^i_t$ such that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\eta \mathbb{1}_{D_i} \mathbb{1}_{\widehat{F}_t}\right) &= \mathbb{E}_{\mathbb{P}}\left(\eta \mathbb{1}_{D_i} \mathbb{1}_{F^i_t}\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\frac{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right)}{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right)} \mathbb{E}_{\mathbb{P}}\left(\eta \mathbb{1}_{D_i} \,\middle|\, \mathcal{F}^i_t\right) \mathbb{1}_{F^i_t}\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{D_i} \frac{\mathbb{E}_{\mathbb{P}}\left(\eta \mathbb{1}_{D_i} \,\middle|\, \mathcal{F}^i_t\right)}{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right)} \mathbb{1}_{F^i_t}\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{D_i} \frac{\mathbb{E}_{\mathbb{P}}\left(\eta \mathbb{1}_{D_i} \,\middle|\, \mathcal{F}^i_t\right)}{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}^i_t\right)} \mathbb{1}_{\widehat{F}_t}\right)
\end{aligned}$$

where the first and last equality follow from the definition of $\widehat{\mathcal{F}}_t$. □

Example 6.3.1. As a simple check of equation (6.9), let us suppose $\Omega = D_1 \cup D_2$ and the σ-algebras $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ are trivial. Then

$$\widehat{\mathcal{F}}_0 = \{\{\phi\}, \{\Omega\}, D_1, D_2\}.$$

For any integrable random variable η, we have

$$\mathbb{E}_{\mathbb{P}}\left(\eta \,\middle|\, \widehat{\mathcal{F}}_0\right) = \mathbb{1}_{D_1} \frac{\mathbb{E}_{\mathbb{P}}(\eta \mathbb{1}_{D_1})}{\mathbb{E}_{\mathbb{P}}(D_1)} + \mathbb{1}_{D_2} \frac{\mathbb{E}_{\mathbb{P}}(\eta \mathbb{1}_{D_2})}{\mathbb{E}_{\mathbb{P}}(D_2)},$$

which agrees with the result of Lemma 6.3.3.

Lemma 6.3.4. *The filtration $\widehat{\mathbb{F}}$ is right-continuous.*

Proof. It is easy to see that $(\widehat{\mathcal{F}}_t)_{t\geq 0}$ is an increasing family of σ-algebras. Therefore, we only need to check the right-continuity.

Suppose $A \in \cap_{s>t}\widehat{\mathcal{F}}_s$, we show that $\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_A \,\middle|\, \widehat{\mathcal{F}}_t\right) = \mathbb{1}_A$. By linearity of conditional expectations, we only need to work with $A \cap D_i$, which has the representation

$$A \cap D_i = \lim_{q \downarrow t} A_q^i \cap D_i$$

where $A_q^i \in \mathcal{F}_q^i$ for $i = 1, \ldots, k$. Using Lemma 6.3.3 and the dominated convergence theorem, we obtain

$$\begin{aligned}
\mathbb{P}\left(\lim_{q\downarrow t} A_q^i \cap D_i \,\middle|\, \widehat{\mathcal{F}}_t\right) &= \lim_{q\downarrow t} \frac{\mathbb{P}\left(A_q^i \cap D_i \,\middle|\, \mathcal{F}_t^i\right)}{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}_t^i\right)} \mathbb{1}_{D_i} \\
&= \lim_{q\downarrow t} \lim_{s\downarrow t} \frac{\mathbb{P}\left(A_q^i \cap D_i \,\middle|\, \mathcal{F}_s^i\right)}{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}_t^i\right)} \mathbb{1}_{D_i} \\
&= \lim_{q\downarrow t} \mathbb{1}_{A_q^i} \mathbb{1}_{D_i}
\end{aligned}$$

where the second equality holds by the right-continuity of $\mathbb{F}^i$ and the last equality holds by picking $s > q$. □

Lemma 6.3.5. *Let ξ be a bounded predictable $\widehat{\mathbb{F}}$-adapted process. Then, for every $i = 1, \ldots, k$, there exists an $\mathbb{F}^i$-predictable process ξ^i such that*

$$\xi = \sum_{i=1}^{n} \xi^i \mathbb{1}_{D_i} \tag{6.10}$$

and the processes ξ^i, $i = 1, \ldots, k$ can be bounded by the same constant as ξ.

Proof. The $\widehat{\mathbb{F}}^\tau$-predictable σ-algebra $\mathcal{P}(\widehat{\mathbb{F}})$ is generated by

$$P(\widehat{\mathbb{F}}) = \{A \times \{0\} \,:\, A \in \widehat{\mathcal{F}}_0\} \cup \{A \times]s,t] \;: 0 < s < t,\; s,t \in \mathbb{Q}_+,\; A \in \bigvee_{r<s} \widehat{\mathcal{F}}_r\}.$$

We need only to prove the claim of the lemma for generators of $\mathcal{P}(\widehat{\mathbb{F}})$. For the family of sets $\{A \times \{0\} \,:\, A \in \widehat{\mathcal{F}}_0\}$ with $A \in \mathcal{F}_0^\tau$, we have, by Lemma 6.3.3,

$$A\mathbb{1}_{\{0\}} = \sum_{i=1}^{n} A_0^i \mathbb{1}_{\{0\}} \mathbb{1}_{D_i}$$

and thus we may set $\xi_t^i := A_0^i \mathbb{1}_{\{0\}}$. Again by Lemma 6.3.3, any bounded, $\widehat{\mathcal{F}}_s$-measurable random variable A_s has the form $A_s = \sum_{i=1}^n A_s^i \mathbb{1}_{D_i}$ where $A_s^i \in \mathcal{F}_s^i$. The generator of $\mathcal{P}(\widehat{\mathbb{F}})$ takes the form

$$A_s \mathbb{1}_{]s,t]} = \sum_{i=1}^{n} A_s^i \mathbb{1}_{]s,t]} \mathbb{1}_{D_i}.$$

It is obvious that $\mathbb{1}_{]s,t]}$ is a left-continuous process in t. All that remains to see is that for each i the process A_s^i is $\mathbb{F}^i$-measurable which implies $A_s^i \mathbb{1}_{]s,t]} \in \mathcal{P}(\mathbb{F}^i)$. Therefore, it is sufficient to take $\xi_t^i := A_s^i \mathbb{1}_{]s,t]}$. □

Corollary 6.3.1. *For every $i = 1, \ldots, n$, the process ξ^i in (6.10) takes the form, $\xi^i = \frac{{}^{p,\mathbb{F}^i}(\xi \mathbb{1}_{D_i})}{{}^{p,\mathbb{F}^i}(\mathbb{1}_{D_i})}$.*

Proof. It is enough to observe $\xi \mathbb{1}_{D_i} = \xi^i \mathbb{1}_{D_i}$ and thus

$$
{}^{p,\mathbb{F}^i}(\xi \mathbb{1}_{D_i}) = {}^{p,\mathbb{F}^i}(\xi^i \mathbb{1}_{D_i}) = {}^{p,\mathbb{F}^i}(\mathbb{1}_{D_i})\xi^i,
$$

since the process ξ^i is $\mathbb{F}^i$-predictable. □

To show that the hypothesis (H') holds between $\mathbb{F}$ and $\widehat{\mathbb{F}}$, we follow the idea of Protter [96].

Proposition 6.3.3. *If for every $i = 1, \ldots, k$, the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^i$, then the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}$.*

Proof. Given a $\mathbb{F}$-semimartingale X, we will prove the claim by contradiction. Suppose that X is not an $\widehat{\mathbb{F}}$-semimartingale. Then there exists $t \geq 0$, $\epsilon > 0$ and a sequence of simple $\widehat{\mathbb{F}}$-predictable process $\{\xi^n\}_{n\in\mathbb{N}}$ converging uniformly to zero, such that

$$
\inf_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}} \left(1 \wedge \left|(\xi^j \bullet X)_t\right|\right) \geq \epsilon. \tag{6.11}
$$

Firstly, the process $(\xi_s^n)_{s\geq 0}$ is bounded $\widehat{\mathbb{F}}$-predictable. Direct computations give

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}} \left(1 \wedge \left|(\xi^n \bullet X)_t\right|\right) &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\big(1 \wedge \big|(\sum_{i=1}^k \xi^{n,i} \mathbb{1}_{D_i} \bullet X)_t\big|\big) \\
&\leq \lim_{n\to\infty} \sum_{i=1}^k \mathbb{E}_{\mathbb{P}}\big(1 \wedge \big|(\xi^{n,i} \mathbb{1}_{D_i} \bullet X)_t\big|\big) \\
&\leq \lim_{n\to\infty} \sum_{i=1}^k \mathbb{E}_{\mathbb{P}}\big(1 \wedge \big|(\xi^{n,i} \bullet X)_t\big|\big) = 0
\end{aligned}
$$

where the limit is zero, since for all $i = 1, \ldots, k$ the process X is assumed to be a $\mathbb{F}^i$-semimartingale. This contradicts (6.11) and thus shows X is an $\widehat{\mathbb{F}}$-semimartingale. □

Since we now know that the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}$, the next step is to compute the $\widehat{\mathbb{F}}$-semimartingale decomposition of $\mathbb{F}$-martingales. As was mentioned, if we assume that $\mathbb{F}^i = \mathbb{F}$ for all $i = 1, \ldots, k$ then we retrieve the semimartingale decomposition result obtained in [84] and [109] for the initial enlargement of filtration.

Lemma 6.3.6. *Assume that M^i is a bounded $\mathbb{F}^i$-martingale, for $i = 1, \ldots, k$. Then the following process*

$$
\widehat{M}_t := \sum_{i=1}^k \mathbb{1}_{D_i} \left(M_0^i - \int_{(0,t]} \frac{1}{N_u^i}\, d[N^i, M]_u \right)
$$

is an $\widehat{\mathbb{F}}$-martingale, where we set, for $i = 1, \ldots, k$,

$$
N_t^i := \mathbb{P}\left(D_i \,\middle|\, \mathcal{F}_t^i\right). \tag{6.12}
$$

Proof. Let us define for $i = 1, \dots, k$, the following $\mathbb{F}^{\tau_i}$ adapted processes

$$B_t^i := -\int_{(0,t]} (N_u^i)^{-1} d[M^i, N^i]_u$$

where N^i are given by (6.12). To show that $\widehat{M}$ is an $\widehat{\mathbb{F}}$-martingale, it is sufficient to show for every $i = 1, \dots, k$,

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{D_i}(M_t^i + B_t^i) \,\middle|\, \widehat{\mathcal{F}}_s\right) = \mathbb{1}_{D_i}(M_s^i + B_s^i). \tag{6.13}$$

To this end, we note that Lemma 6.3.3 yields

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{D_i}(M_t^i + B_t^i) \,\middle|\, \widehat{\mathcal{F}}_s\right) = \mathbb{1}_{D_i} \frac{\mathbb{E}_{\mathbb{P}}\left((M_t^i + B_t^i)\mathbb{1}_{D_i} \,\middle|\, \mathcal{F}_s^i\right)}{\mathbb{P}\left(D_i \,\middle|\, \mathcal{F}_s^i\right)}$$

and we proceed by computing the numerator of the right-hand side. We obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left((M_t^i + B_t^i)\mathbb{1}_{D_i} \,\middle|\, \mathcal{F}_s^i\right) &= \mathbb{E}_{\mathbb{P}}\left((M_t^i + B_t^i)N_t^i \,\middle|\, \mathcal{F}_s^i\right) \\
&= (M_s^i + B_s^i)N_s^i + \int_{(s,t]} N_u^i dB_u^i + \int_{(s,t]} d[M^i, N^i]_u \\
&= (M_s^i + B_s^i)N_s^i
\end{aligned}$$

where the second equality holds by integration parts formula. Therefore, equality (6.13) is valid and the claim of the lemma follows. □

Remark 6.3.2. As a special case, if the immersion holds between $\mathbb{F}$ and $\mathbb{F}^i$ for every $i = 1, \dots, k$, then M is an $\mathbb{F}$-martingale and

$$\widehat{M}_t := M_t - \sum_{i=1}^{k} \mathbb{1}_{D_i} \left(\int_{(0,t]} \frac{1}{N_u^i} \, d[N^i, M]_u \right)$$

is an $\widehat{\mathbb{F}}$-martingale.

In the following, we apply the above results to show that the hypothesis (H') is closed under minimum and maximum. The idea is to decompose the space Ω into $\{\tau_1 \leq \tau_2\} \cup \{\tau_1 > \tau_2\}$ and define the following family of sets

$$\widehat{\mathcal{F}}_t^{\tau_1 \wedge \tau_2} := \{A \in \mathcal{G} \,|\, \exists A_t^{\tau_1} \in \mathcal{F}_t^{\tau_1}, \exists A_t^{\tau_2} \in \mathcal{F}_t^{\tau_2} \text{ s.t. } A = (A_t^{\tau_2} \cap \{\tau_1 > \tau_2\}) \cup (A_t^{\tau_1} \cap \{\tau_1 \leq \tau_2\})\}$$

$$\widehat{\mathcal{F}}_t^{\tau_1 \vee \tau_2} := \{A \in \mathcal{G} \,|\, \exists A_t^{\tau_1} \in \mathcal{F}_t^{\tau_1}, \exists A_t^{\tau_2} \in \mathcal{F}_t^{\tau_2} \text{ s.t. } A = (A_t^{\tau_1} \cap \{\tau_1 > \tau_2\}) \cup (A_t^{\tau_2} \cap \{\tau_1 \leq \tau_2\})\}$$

for every $t \geq 0$.

Lemma 6.3.7. *The families $\widehat{\mathbb{F}}^{\tau_1 \vee \tau_2} := (\widehat{\mathcal{F}}_t^{\tau_1 \vee \tau_2})_{t \geq 0}$ and $\widehat{\mathbb{F}}^{\tau_1 \wedge \tau_2} := (\widehat{\mathcal{F}}_t^{\tau_1 \wedge \tau_2})_{t \geq 0}$ are right-continuous filtrations.*

Proof. For $i = 1, 2$, we set $\mathbb{F}^i = \mathbb{F}^{\tau_i}$. For the case of $\widehat{\mathbb{F}}^{\tau_1 \wedge \tau_2}$, we set $D_1 = \{\tau_1 \leq \tau_2\}$ and $D_2 = \{\tau_1 > \tau_2\}$. For the case of $\widehat{\mathbb{F}}^{\tau_1 \vee \tau_2}$, we set $D_1 = \{\tau_1 > \tau_2\}$ and $D_2 = \{\tau_1 \leq \tau_2\}$. It is then sufficient to apply Lemma 6.3.4 □

Corollary 6.3.2. *(i) The filtration $\mathbb{F}^{\tau_1 \wedge \tau_2}$ is strictly contained in $\widehat{\mathbb{F}}^{\tau_1 \wedge \tau_2}$.*
(ii) the filtration $\mathbb{F}^{\tau_1 \vee \tau_2}$ is strictly contained in $\widehat{\mathbb{F}}^{\tau_1 \vee \tau_2}$.

Proof. It is easy to see that $\{\tau_1 \wedge \tau_2 > t\} \in \widehat{\mathcal{F}}_t^{\tau_1 \wedge \tau_2}$, therefore $\tau_1 \wedge \tau_2$ is a $\widehat{\mathbb{F}}^{\tau_1 \wedge \tau_2}$-stopping time. Recall that $\mathbb{F}^{\tau_1 \wedge \tau_2}$ is, by definition, the smallest right-continuous filtration for which $\tau_1 \wedge \tau_2$ is a stopping time. Hence, by Lemma 6.3.7, the filtration $\mathbb{F}^{\tau_1 \wedge \tau_2}$ is contained in $\widehat{\mathbb{F}}^{\tau_1 \wedge \tau_2}$. To see that the inclusion is strict, we notice that the set $\{\tau_1 \leq \tau_2\}$ is an element of $\widehat{\mathcal{F}}_t^{\tau_1 \wedge \tau_2}$ for all $t \geq 0$, but not an element of $\mathcal{F}_t^{\tau_1 \wedge \tau_2}$ for all $t \geq 0$. A similar argument applies to $\tau_1 \vee \tau_2$. □

Lemma 6.3.8. *If τ_1 and τ_2 are two random times satisfying hypothesis (H') then:*
(i) the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}^{\tau_1\wedge\tau_2}$.
(ii) the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}^{\tau_1\vee\tau_2}$.

Proof. It is sufficient to apply Proposition 6.3.3. □

Corollary 6.3.3. *If τ_1 and τ_2 are two random times satisfying hypothesis (H') then:*
(i) the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^{\tau_1\wedge\tau_2}$,
(ii) the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^{\tau_1\vee\tau_2}$.

Proof. By Corollary 6.3.2, the filtration $\mathbb{F}^{\tau_1\wedge\tau_2}$ (resp. $\mathbb{F}^{\tau_1\vee\tau_2}$) is a sub-filtration of $\widehat{\mathbb{F}}^{\tau_1\wedge\tau_2}$ (resp. $\widehat{\mathbb{F}}^{\tau_1\vee\tau_2}$) and from Lemma 6.3.8, the hypothesis (H') is satisfied between $\mathbb{F}$ and $\widehat{\mathbb{F}}^{\tau_1\wedge\tau_2}$ (resp. $\widehat{\mathbb{F}}^{\tau_1\vee\tau_2}$). Therefore, the claims follow from Lemma 6.3.1. □

Remark 6.3.3. From Lemma 6.3.8, we notice that if τ_1 and τ_2 are two random times satisfying the hypothesis (H) then, in general, $\tau_1\wedge\tau_2$ may not satisfy the hypothesis (H). More explicitly, the $\mathbb{F}^{\tau_1\wedge\tau_2}$-semimartingale decomposition of a $\mathbb{F}$-martingale M is given by

$$M_t := \mathbb{E}_{\mathbb{P}}\left(M_t^{\widehat{\mathbb{F}}^{\tau_1\wedge\tau_2}} \,\middle|\, \mathcal{F}_t^{\tau_1\wedge\tau_2}\right) + \mathbb{E}_{\mathbb{P}}\left(\sum_{i=1}^{k} \mathbb{1}_{D_i}\left(\int \frac{1}{N_u^i}[N^i, M]_u\right) \,\middle|\, \mathcal{F}_t^{\tau_1\wedge\tau_2}\right)$$

where $D_1 = \{\tau_1 \le \tau_2\}$, $D_2 = D_1^c$ and $\mathbb{E}_{\mathbb{P}}\left(M^{\widehat{\mathbb{F}}^{\tau_1\wedge\tau_2}} \,\middle|\, \mathcal{F}^{\tau_1\wedge\tau_2}\right)$ is an $\mathbb{F}^{\tau_1\wedge\tau_2}$-martingale. It is easy to see that if the random times τ_1 and τ_2 are ordered then the second term in the right-hand side vanishes.

6.4 Pseudo-Stopping Time and Min and Max

As usual, we denote by $\mathbb{F}^\tau$ the progressive enlargement of $\mathbb{F}$ with a random time τ. The main goal of this section is to examine the class of $\mathbb{F}$-pseudo-stopping times. Let us remind that when we say that τ satisfies the hypothesis (H), we mean that the filtration $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$. The following definition was introduced in Nikeghbali and Yor [92].

Definition 6.4.1. A random time τ is a said be an $\mathbb{F}$-*pseudo-stopping time* if the equality $\mathbb{E}_{\mathbb{P}}(M_\tau) = \mathbb{E}_{\mathbb{P}}(M_0)$ holds for any bounded $\mathbb{F}$-martingale M.

The following lemma is taken from page 186 of [31]. It will be used to give in Lemma 6.4.2 an alternative proof of Theorem 1 in [92].

Lemma 6.4.1. *For any random time τ and any $\mathbb{F}^\tau$-stopping time ρ, there exist an $\mathbb{F}$-stopping stopping T such that $\tau\wedge\rho = \tau\wedge T$.*

Lemma 6.4.2. *A random time τ is a $\mathbb{F}$-pseudo-stopping time if and only if $M_{\tau\wedge t}$ is an $\mathbb{F}^\tau$-martingale for any bounded $\mathbb{F}$-martingale M.*

Proof. Suppose that $M_{\tau\wedge t}$ is a bounded $\mathbb{F}^\tau$-martingale. Then, by the dominated convergence theorem, we obtain

$$\mathbb{E}_{\mathbb{P}}(M_\tau) = \lim_{t\to\infty} \mathbb{E}_{\mathbb{P}}(M_{\tau\wedge t})$$

which is equal to $\mathbb{E}_{\mathbb{P}}(M_0)$ by the fact that $M_{\tau\wedge t}$ is an $\mathbb{F}^\tau$-martingale. Therefore, τ is an $\mathbb{F}$-pseudo-stopping time. To prove the converse, for every bounded $\mathbb{F}^\tau$-stopping time ρ, by Lemma 6.4.1, there exists an $\mathbb{F}$-stopping time T such that $\tau\wedge\rho = \tau\wedge T$. Therefore,

$$\mathbb{E}_{\mathbb{P}}(M_{\tau\wedge\rho}) = \mathbb{E}_{\mathbb{P}}(M_{\tau\wedge T}) = \mathbb{E}_{\mathbb{P}}\left(\widehat{M}_\tau\right)$$

where $\widehat{M}_u := M_{u\wedge T}$ is a bounded $\mathbb{F}$-martingale. Since τ is an $\mathbb{F}$-pseudo-stopping time, we have

$$\mathbb{E}_{\mathbb{P}}\left(\widehat{M}_\tau\right) = \mathbb{E}_{\mathbb{P}}\left(\widehat{M}_0\right) = \mathbb{E}_{\mathbb{P}}\left(M_0\right)$$

and $(M_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{F}^\tau$-martingale. □

An $\mathbb{F}$-pseudo-stopping time does not in general satisfy the hypothesis (H). However, it is known from [92] that if a random time τ satisfies the hypothesis (H) then it is a pseudo-stopping time and it satisfies trivially the hypothesis (HP). This observation leads to the next lemma, where we establish the partial converse to the above mentioned implication.

Lemma 6.4.3. *Assuming that all $\mathbb{F}$-martingales are continuous then the following are equivalent:*
(i) the random time τ is an $\mathbb{F}$-pseudo-stopping time and it satisfies the hypothesis (HP),
(ii) the random time τ satisfies the hypothesis (H).

Proof. We only need to show that (i) implies (ii). Since all $\mathbb{F}$-martingales are continuous, the optional and predictable σ-algebras coincide. Therefore, the process $F_{u,t}$ is a continuous martingale for $t \geq u$. We also know, from Theorem 1 in [92], that the process $F_t := \mathbb{P}\left(\tau \leq t \,\middle|\, \mathcal{F}_t\right)$ is non-decreasing and predictable. Furthermore, since the random time τ also satisfies the hypothesis (HP), for every u, the process

$$C_{u,t} = \frac{F_{u,t}}{F_t}, \quad t \geq u,$$

is decreasing in t. Therefore, for every $t \geq u$, the process

$$F_{u,t} = F_t C_{u,t}$$

is a continuous martingale of finite variation and thus $F_{u,t} = F_{u,u}$. □

The following result is borrowed from [27] (see Lemma 4.6 therein).

Lemma 6.4.4. *Let τ be a random time which is an $\mathcal{F}_\infty$-measurable random variable. Then the set of probability measures equivalent to $\mathbb{P}$ for which the hypothesis (H) holds is non-empty if and only if τ is an $\mathbb{F}$-stopping time (in this case, the equality $\mathbb{F}^\tau = \mathbb{F}$ holds).*

Corollary 6.4.1. *Given a random time τ and suppose all $\mathbb{F}$-martingales are continuous then the following are equivalent:*
(i) τ is an honest time and an $\mathbb{F}$-pseudo-stopping time,
(ii) τ is an $\mathbb{F}$-stopping time.

Proof. We only need to show (ii) follows from (i), as the converse is obvious. It is known that an honest time satisfies the hypothesis (HP). Therefore, by Lemma 6.4.3, the random time τ satisfies the hypothesis (H) and the claim then follows from Lemma 6.4.4, since an honest times is $\mathcal{F}_\infty$-measurable. □

Proposition 6.4.1. *Let τ and ρ be two $\mathbb{F}$-pseudo-stopping times. Then $\tau\vee\rho$ is an $\mathbb{F}$-pseudo-stopping time if and only if $\tau\wedge\rho$ is an $\mathbb{F}$-pseudo-stopping time.*

Proof. We observe that the following relationship holds

$$M_{\rho\vee\tau} + M_{\rho\wedge\tau} = M_\rho + M_\tau.$$

This immediately shows that

$$\mathbb{E}_{\mathbb{P}}\left(M_{\tau\vee\rho}\right) + \mathbb{E}_{\mathbb{P}}\left(M_{\rho\wedge\tau}\right) = \mathbb{E}_{\mathbb{P}}\left(M_\rho + M_\tau\right) = 2\mathbb{E}_{\mathbb{P}}\left(M_0\right),$$

which concludes the proof. □

Let us remind the reader that for an arbitrary filtration, the dual $\mathbb{K}$-optional projection of $\mathbb{1}_{[\![\tau,\infty[\![}$ is denoted by $\bar{A}^{\tau,\mathbb{K}}$.

Lemma 6.4.5. *Let τ be a random time and ρ an $\mathbb{F}$-pseudo-stopping time. Then $\tau\wedge\rho$ is an $\mathbb{F}$-pseudo-stopping time if and only if the equality*

$$\mathbb{E}_{\mathbb{P}}\left(M_\rho \bar{A}_\infty^{\tau,\mathbb{F}^\rho}\right) = \mathbb{E}_{\mathbb{P}}\left(M_\rho\right) \tag{6.14}$$

holds for all bounded $\mathbb{F}$-martingales M. The process $A^{\tau,\mathbb{F}^\rho}$ is the $\mathbb{F}^\rho$-dual optional projection of $\mathbb{1}_{\{\tau\le t\}}$.

Proof. We compute directly $\mathbb{E}_{\mathbb{P}}\left(M_{\tau\wedge\rho}\right)$. Using the properties of the dual predictable projection, we obtain

$$\mathbb{E}_{\mathbb{P}}\left(M_{\tau\wedge\rho}\right) = \mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty]} M_{\rho\wedge u}\, d\bar{A}_u^{\tau,\mathbb{F}^\rho}\right)$$

and, by Theorem 1 of [92], the process $\widehat{M}_u := M_{\rho\wedge u}$ is a bounded $\mathbb{F}^\rho$-martingale. Therefore,

$$\mathbb{E}_{\mathbb{P}}\left(\int_{[0,\infty]} M_{\rho\wedge u}\, d\bar{A}_u^{\tau,\mathbb{F}^\rho}\right) = \mathbb{E}_{\mathbb{P}}\left(\widehat{M}_\infty \int_{[0,\infty]} d\bar{A}_u^{\tau,\mathbb{F}^\rho}\right) = \mathbb{E}_{\mathbb{P}}\left(\widehat{M}_\infty \bar{A}_\infty^{\tau,\mathbb{F}^\rho}\right) = \mathbb{E}_{\mathbb{P}}\left(M_\rho \bar{A}_\infty^{\tau,\mathbb{F}^\rho}\right)$$

and thus $\tau\wedge\rho$ is an $\mathbb{F}$-pseudo-stopping time if and only if for every bounded $\mathbb{F}$-martingale M the equality

$$\mathbb{E}_{\mathbb{P}}\left(M_\rho \bar{A}_\infty^{\tau,\mathbb{F}^\rho}\right) = \mathbb{E}_{\mathbb{P}}\left(M_\rho\right)$$

holds. □

Corollary 6.4.2. *If ρ is an $\mathbb{F}$-pseudo-stopping time and τ is an $\mathbb{F}^\rho$-pseudo-stopping time then $\tau\wedge\rho$ is an $\mathbb{F}$-pseudo-stopping time.*

Proof. If τ is an $\mathbb{F}^\rho$-pseudo-stopping time then $\bar{A}_\infty^{\tau,\mathbb{F}^\rho} = 1$ and the claim follows from Lemma 6.4.5. □

Corollary 6.4.2 can be seen as a generalization of Proposition 4 in [92] where τ was assumed to be an $\mathbb{F}^\rho$-stopping time.

Chapter 7

Admissibility of Generic Market Models of Forward Swaps Rates

This chapter is based on the paper with the same title by Li and Rutkowski [76] accepted for publication in *Mathematical Finance*.

7.1 Introduction

Our main goal is to re-examine the *admissibility problem* for an arbitrary family of forward swaps. This issue was previously studied in Galluccio et al. [42] (see also Pietersz and Regenmortel [95] for the related study of a special case) who attempted to provide necessary and sufficient conditions for a family of forward swaps to have the ability to underpin an arbitrage-free model of the term structure of interest rates with a finite tenor structure. Additionally, we will also examine an extension of this problem to the situation when the payments dates of forward swaps are chosen to be only some of the tenor structure dates between the start date of a swap and its maturity date. This generalization is motivated by the case where some forward swaps (typically, of shorter maturities) have quarterly payments, whereas for some other swaps (with longer maturities) the payments are exchanged according to the semi-annual schedule.

Let $\mathcal{T} = \{T_0, \ldots, T_n\}$ with $0 < T_0 < T_1 < \cdots < T_{n-1} < T_n$ be a fixed sequence of dates representing the complete *tenor structure* for all forward swaps under consideration. For every $i = 1, \ldots, n$, we write $a_i = T_i - T_{i-1}$ to denote the length of the ith accrual period. Let $B(t, T_i)$ stand for the price of the *unit zero-coupon bond* maturing at T_i. Unless stated otherwise, it is assumed that for every $i = 0, \ldots, n$ the bond price $B(t, T_i)$, $t \in [0, T_i]$, is governed by a positive stochastic process with the obvious terminal condition $B(T_i, T_i) = 1$.

Let $\mathcal{S} = \{S_1, \ldots, S_l\}$ be an arbitrary family of l distinct fixed-for-floating forward swaps associated with the tenor structure $\mathcal{T}$, in the sense that any reset or settlement date for any swap S_j in $\mathcal{S}$ belongs to $\mathcal{T}$. For concreteness, we assume throughout that the frequencies of fixed and floating payments are the same. As commonly assumed, we postulate that the floating leg payments are specified by the level of the LIBOR at each reset date and payments are exchanged at the next settlement date. The *forward swap rate* κ^j for the standard forward swap S_j, which starts at T_{s_j} and matures at T_{m_j}, is well known to be given by the following expression (see, for instance, Section 13.1 in Musiela and Rutkowski [90])

$$\kappa_t^j = \kappa_t^{s_j, m_j} := \frac{B(t, T_{s_j}) - B(t, T_{m_j})}{\sum_{i=s_j+1}^{m_j} a_i B(t, T_i)} = \frac{P_t^{s_j, m_j}}{A_t^{s_j, m_j}}, \quad \forall\, t \in [0, T_{s_j}], \tag{7.1}$$

where we denote

$$P_t^{s_j, m_j} := B(t, T_{s_j}) - B(t, T_{m_j}), \quad \forall\, t \in [0, T_{s_j}], \tag{7.2}$$

and

$$A_t^{s_j,m_j} := \sum_{i=s_j+1}^{m_j} a_i B(t,T_i), \quad \forall\, t \in [0, T_{s_j+1}]. \tag{7.3}$$

The process A^{s_j,m_j} is usually called the *swap annuity* or the *swap numéraire* for the forward swap S_j, although some authors prefer the term *level process*. In practical applications, it is also referred to as the present value of the basis point of the swap. The dates $T_{s_j}, T_{s_j+1}, \dots, T_{m_j-1}$ are then the *reset* dates for the forward swap S_j, whereas the dates $T_{s_j+1}, T_{s_j+2}, \dots, T_{m_j}$ are *settlement* dates. Therefore, the collection of dates $T_{s_j}, T_{s_j+1}, \dots, T_{m_j}$ represents the *reset/settlement* dates in the forward swap S_j. It is a simple observation that these dates are the only *relevant dates* for the (standard) forward swap S_j.

Remark 7.1.1. In what follows, we will consider a more general set-up, in which the start date T_{s_j} is the first reset date and the maturity date T_{m_j} is the last settlement date in the forward swap S_j. Other reset/settlement dates in a forward swap S_j can be chosen freely from the set of dates $T_i \in \mathcal{T}$ that satisfy $T_{s_j} < T_i < T_{m_j}$. Then all reset/settlement dates in the forward swap S_j will be simply referred to as the *relevant dates* in S_j. They can also be seen as the tenor structure *generated* by the forward swap S_j; this observation justifies the use of the symbol $\mathcal{T}(S_j)$. In the general case, we write

$$\kappa_t^j = \kappa_t^{s_j,m_j} = \frac{B(t,T_{s_j}) - B(t,T_{m_j})}{\sum_{i\in\mathcal{A}_j} \widetilde{a}_i B(t,T_i)} = \frac{P_t^{s_j,m_j}}{A_t^{s_j,m_j}}, \quad \forall\, t \in [0, T_{s_j}], \tag{7.4}$$

where $\mathcal{A}_j$ is the set of indices of all settlements dates in the swap S_j and $\widetilde{a}_i$ represents the ith adjusted accrual period. For instance, if a forward swap S_j has only one settlement date, specifically, its maturity date, then $\mathcal{T}(S_j) = \{T_{s_j}, T_{m_j}\}$ and $\mathcal{A}_j = \{m_j\}$. Furthermore, equation (7.4) becomes

$$\kappa_t^j = \frac{B(t,T_{s_j}) - B(t,T_{m_j})}{\widetilde{a}_{m_j} B(t,T_{m_j})}$$

where $\widetilde{a}_{m_j} = T_{m_j} - T_{s_j} = \sum_{i=s_j+1}^{m_j} a_i$; this corresponds to the forward LIBOR (see equation (7.10)) If all the dates from $\mathcal{T}$ between the start and maturity dates are the settlement dates for a given forward swap, the contract is referred to as a *standard* forward swap.

Let us select the bond maturing at some date $T_b \in \mathcal{T}$ as a *bond numéraire* and let us denote by $\mathcal{B}^b$ the family of the *deflated bond prices*

$$\mathcal{B}^b := \big\{ B^b(t,T_i) = B(t,T_i)/B(t,T_b),\ i = 0, \dots, n \big\}.$$

In terms of the deflated bond prices $B^b(t,T_i)$, for the standard forward swap we obtain

$$\kappa_t^{s_j,m_j} = \frac{B^b(t,T_{s_j}) - B^b(t,T_{m_j})}{\sum_{i=s_j+1}^{m_j} a_i B^b(t,T_i)} = \frac{P_t^{b,s_j,m_j}}{A_t^{b,s_j,m_j}}, \quad \forall\, t \in [0, T_{s_j} \wedge T_b], \tag{7.5}$$

where we write

$$P_t^{b,s_j,m_j} = B^b(t,T_{s_j}) - B^b(t,T_{m_j}), \quad \forall\, t \in [0, T_{s_j} \wedge T_b], \tag{7.6}$$

and

$$A_t^{b,s_j,m_j} = \sum_{i=s_j+1}^{m_j} a_i B^b(t,T_i), \quad \forall\, t \in [0, T_{s_j+1} \wedge T_b]. \tag{7.7}$$

We call the process A^{b,s_j,m_j} the *deflated swap annuity* or *deflated swap numéraire*. As we shall see in what follows, the (deflated) swap numéraires are crucial objects in the specification of the Radon-Nikodým densities for *swap measures* under which forward swap rates are (local) martingales.

Remark 7.1.2. Note that the choice of a bond numéraire is arbitrary, in the sense that one may take the bond price $B(t,T_b)$ for any maturity date $T_b \in \mathcal{T}$ to play this role. It will be rather clear that the results presented in the sequel do not depend on a particular choice of a bond numéraire, but only provided that all bond prices are either assumed or shown to follow positive processes.

We are in a position to describe the main problems considered in this work. We are interested in deriving the joint dynamics of forward swap rates associated with a family $\mathcal{S} = \{S_1, \dots, S_l\}$ of forward swaps, as well as in checking whether these dynamics are supported by the existence of a (unique) family of deflated bond prices $\mathcal{B}^b = \{B^b(t, T_i),\ i = 1, \dots, n\}$ for some choice (or all choices) of a bond numéraire.

We will thus deal with two related problems, which can be described as follows.

- Under which assumptions, a given swap rate model can be supported by the existence of an arbitrage-free term structure model consistent with swap rates, where by a term structure model we mean the joint dynamics of positive deflated bond prices?

- Under which assumptions, the joint dynamics for a given family of forward swaps is uniquely specified under a single probability measure in terms of 'drifts' and 'volatilities'?

The answers to the above two questions will be examined in Sections 7.2 and 7.3, respectively. In both cases, the technique is to formulate a suitable *inverse problem* and to examine its solvability. For the first question, we are able to provide both necessary conditions and sufficient conditions for the existence of a unique and positive solution to the inverse problem for deflated bond prices (see Inverse Problem (IP.1)), that is, for the (weak) $\mathcal{T}$-admissibility of a family $\mathcal{S}$ (see Definition 7.2.6 and Propositions 7.2.2, 7.2.7 and 7.2.8).

It is worth stressing that our conditions differ from these provided in Galluccio et al. [42], although it should also be acknowledged that a thorough analysis of their work was the original motivation for this research. In fact, we show by means of counter-examples that the main result in [42] is defective and thus the present research could also be seen as a partial correction to this result. For the second question, we introduce the corresponding inverse problem for swap annuities (see Inverse Problem (IP.2)) and the concept of the $\mathcal{A}$-admissibility (see Definition 7.3.2). It will be shown that the (weak) $\mathcal{T}$-admissibility implies the (weak) $\mathcal{A}$-admissibility, but the converse does not hold. We then describe in Section 7.3.2 a general methodology for the computation of the joint dynamics of forward swap rates, which covers all potentially interesting $\mathcal{A}$-admissible cases. For the sake of generality, we examine the set-up of a model driven by semimartingales; for most practical purposes this abstract framework can be easily specified to the case of a model driven by a multi-dimensional correlated Brownian motions (or jump-diffusion processes).

It should be acknowledged that that research presented in this work is by no means conclusive and in fact several related problems remain still open. Firstly, various market models similar to forward swap rate models can also be formally derived for certain families of forward CDS spreads and LIBORs. For preliminary studies of market models of this kind, the interested reader is referred to Brigo [23] and Li and Rutkowski [73]. To the best of our knowledge, the issue of admissibility of a joint family of forward swap rates and forward CDS spreads was not systematically examined in the existing literature, however, and thus this issue remains largely open.

Secondly, and more importantly, the recent credit crisis has changed to a large extent the way in which practitioners discount future payoffs. The revised approach to the discounting convention resulted also in an essential change in the way in which valuation of interest rate derivatives is performed and, as a consequence, it also reignited interest in term structure modeling. The standard formulae considered in this work can be extended to cover the new perception of the market, as explained in recent papers by Bianchetti [9, 10], Bianchetti and Carlicchi [11], Fujii et al. [40, 41], Kijima et al. [72], Mercurio [80], Moreni and Pallavicini [87], and Pallavicini and Tarenghi [94]. According to the new pricing paradigm for fixed-for-floating interest rate swaps and other related contracts such as swaptions, a single discounting curve used in the traditional approach should be replaced by (at least) two yield curves: the first one is used to specify the level of future floating cash flows, whereas the second one (a proxy for the risk-free interest rate) is used for discounting of fixed and floating cash flows. According to this extended swap valuation methodology, formula

(7.4) should be modified as follows (see, for instance, formula (9) in [72] or page 15 in [80])

$$\widehat{\kappa}^j_t = \widehat{\kappa}^{s_j,m_j}_t := \frac{\sum_{i\in\mathcal{A}_j} \widetilde{a}_i \widehat{B}(t,T_i)\widetilde{L}_i(t)}{\sum_{i\in\mathcal{A}_j} \widetilde{a}_i \widehat{B}(t,T_i)} \tag{7.8}$$

where $\widetilde{L}_i(t)$ represents the forward LIBOR over the relevant accrual period, as computed from the risky LIBOR curve, and $\widehat{B}(t,T_i)$ is a judiciously chosen market risk-free yield curve, such as the OIS (Overnight Index Swap) yield curve. The OIS is a fixed-for-floating interest rate swap which pays periodically the daily compounded overnight rate, instead of the LIBOR. Since the overnight (collateral) rate can be seen as risk-free, the forward swap rate corresponding to the OIS is still given by expression (7.4), where $B(t,T_i)$ is interpreted as the price of a risk-free (for instance, collateralized) zero-coupon bond. For more details on the choice of relevant market rates for each currency, the interested reader may consult, for instance, Fujii et al. [40, 41] or Moreni and Pallavicini [87]. Regarding the LIBOR, one can make the modeling postulate that the forward LIBOR $\widetilde{L}_i$, as seen at time t for the period $[T_{i-1},T_i]$, can be formally given by the following expression

$$\widetilde{L}_i(t) = \frac{\widetilde{B}(t,T_{i-1}) - \widetilde{B}(t,T_i)}{\widetilde{a}_i \widetilde{B}(t,T_i)} \tag{7.9}$$

where the quantities $\widetilde{B}(t,T_i)$, $i=0,\dots,n$, are aimed to represent the implicit LIBOR yield curve. It is worth noting that (7.4) can also be represented as follows

$$\kappa^j_t = \kappa^{s_j,m_j}_t = \frac{\sum_{i\in\mathcal{A}_j} \widetilde{a}_i B(t,T_i)L_i(t)}{\sum_{i\in\mathcal{A}_j} \widetilde{a}_i B(t,T_i)}$$

where the forward LIBOR $L_i(t)$ is given by the traditional pre-crisis formula, that is,

$$L_i(t) = \frac{B(t,T_{i-1}) - B(t,T_i)}{\widetilde{a}_i B(t,T_i)}\,. \tag{7.10}$$

A comparison of (7.4) and (7.8) makes it clear that the two expressions coincide when the equality $\widehat{B}(t,T_i) = B(t,T_i)$ holds and the forward LIBOR is given by (7.10), so that, as expected, the new approach reduces to the classic one when all rates are non-risky. We do not deal with default-risky rates here, so the issues related to admissible families of forward swap rates given by the generic expression (7.8) are left for future research.

7.2 Admissible Families of Forward Swap Rates

In this section, we address the issue whether a given family of forward swap rates is supported by the existence of the implied family of deflated bond prices. This corresponds to the inverse problem examined by Galluccio et al. [42], which can be informally stated as follows:

- Provide necessary and sufficient conditions for a given family of forward swap rates $\mathcal{S} = \{S_1,\dots,S_l\}$ and the corresponding (diffusion-type) swap rate processes $\{\kappa^1,\dots,\kappa^l\}$ to uniquely specify a family of non-zero deflated bond prices $\mathcal{B}^b = \{B^b(t,T_0),\dots,B^b(t,T_n)\}$ for any choice of $b\in\{0,\dots,n\}$.

7.2.1 Linear Systems Associated with Forward Swaps

Let us note that for a given family $\mathcal{B}^n = \{B^n(t,T_i),\, i=0,\dots,n\}$ of stochastic processes representing the deflated bond prices when $B(t,T_n)$ is the numéraire bond, we can uniquely determine the forward swap rate κ^{s_j,m_j} for any start T_{s_j} and maturity T_{m_j} dates from the tenor structure $\mathcal{T}$. This simple observation paves the way for the most commonly used *direct approach* to modeling of forward swap rates, in which one starts by choosing a term structure model and subsequently derives the joint dynamics of a family of forward swap rates of interest.

It is also clear that each forward swap rate κ^{s_j,m_j} generates a linear equation in which deflated bond prices can be seen as 'unknowns'. Specifically, one obtains the following *swap equation* associated with the forward swap S_j and the numéraire bond $B(t,T_b)$

$$- B^b(t,T_{s_j}) + \sum_{i=s_j+1}^{m_j-1} \kappa_t^{s_j,m_j} a_i B^b(t,T_i) + (1+\kappa_t^{s_j,m_j}) a_{m_j} B^b(t,T_{m_j}) = 0. \tag{7.11}$$

In this context, the following *inverse problem* arises in a natural may: describe all families of forward swaps associated with the tenor structure $\mathcal{T}$ such that the corresponding family of forward swap rates uniquely specifies the associated family of non-zero (and preferably strictly positive) deflated bond prices.

To formally address the above-mentioned inverse problem, we identify a forward swap S_j with the corresponding swap equation, in which a generic value κ_j of the forward swap rate κ^{s_j,m_j} plays the role of a parameter. Let us now introduce some notation. For a fixed b, let x_i stand for a generic value of the deflated bond price $B^b(t,T_i)$ and let κ_j be a generic value of the forward swap rate $\kappa_t^j := \kappa_t^{s_j,m_j}$. Since x_i and κ_j are aimed to represent generic values of the corresponding processes in a yet unspecified stochastic model, we have that $(x_0,\dots,x_n) \in \mathbb{R}^{n+1}$ and $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l$, in general.

Since the bond $B(t,T_b)$ is chosen to be the numéraire asset, it is clear, by the definition of the deflated bond price, that the variable x_b satisfies $x_b = 1$ and thus it is more adequate to consider a generic value $(x_0,\dots,x_{b-1},x_{b+1},\dots,x_n) \in \mathbb{R}^n$. Using this notation, we may represent equation (7.5) as follows, for every $j = 1,\dots,l$,

$$\kappa_j = \frac{x_{s_j} - x_{m_j}}{\sum_{i=s_j+1}^{m_j} a_i x_i}. \tag{7.12}$$

For brevity, we write $c_{j,i} = \kappa_j a_i$ and $\widetilde{c}_{j,m_j} = (1+\kappa_j)a_{m_j}$, so that equation (7.11) becomes

$$- x_{s_j} + \sum_{i=s_j+1}^{m_j-1} c_{j,i} x_i + \widetilde{c}_{j,m_j} x_{m_j} = 0. \tag{7.13}$$

Definition 7.2.1. For a given family $\mathcal{S} = \{S_1,\dots,S_l\}$ of forward swaps and any fixed $b \in \{0,\dots,n\}$, the *associated linear system* (7.13) parametrized by a vector $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l$, is denoted as

$$C^b(\kappa_1,\dots,\kappa_l)\overline{x}^b = \overline{e}^b(\kappa_1,\dots,\kappa_l)$$

where $\overline{x}^b = (x_0,\dots,x_{b-1},x_{b+1},\dots,x_n)$ is the vector of unknowns and the vector $\overline{e}^b(\kappa_1,\dots,\kappa_l)$ belongs to $\mathbb{R}^l$.

The matrix $C^b(\kappa_1,\dots,\kappa_l)$ and the vector $\overline{e}^b(\kappa_1,\dots,\kappa_l)$ depend on the vector $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l$. To alleviate notation, we shall frequently suppress the variables $\kappa_1,\dots,\kappa_l$, however, and we will simply write C^b and $\overline{e}^b$.

As already mentioned in Remark 7.1.1, we will also consider more general forward swaps for which the actual payment dates constitutes merely a subset of the collection of all dates between the start and maturity dates. In that case, the sum in the denominator of (7.12) is taken over some dates between T_{s_j+1} and T_{m_j} only. Furthermore, the coefficients a_i are suitably adjusted to represent the length of times between the consecutive payment dates, and a modified version of equation (7.13) is easily derived, so that Definition 7.2.1 covers this general set-up as well. For instance, if a forward swap S_j settles at its maturity date only, then equation (7.12) becomes

$$\kappa_j = \frac{x_{s_j} - x_{m_j}}{\widetilde{a}_{m_j} x_{m_j}}$$

where $\widetilde{a}_j = T_{m_j} - T_{s_j} = \sum_{i=s_j+1}^{m_j} a_i$. The following definition is also tailored to cover the whole spectrum of cases considered in this work.

Definition 7.2.2. A date $T_i \in \mathcal{T}$ is a *relevant date* for the swap S_j whenever $T_i = T_{s_j}$, $T_i = T_{m_j}$, or the term $c_{j,i}$ is non-zero. We write $\mathcal{T}(S_j)$ to denote the set of all relevant dates for the swap S_j and we set $\mathcal{T}_0(S_j) = \{T_{s_j}, T_{m_j}\}$.

The following notation will be useful:
$\mathcal{T}_0(\mathcal{S})$ – the set of all start/maturity dates for the forward swaps from a family $\mathcal{S}$, that is,

$$\mathcal{T}_0(\mathcal{S}) = \bigcup_{j=1}^{l} \mathcal{T}_0(S_j) = \bigcup_{j=1}^{l} \{T_{s_j}, T_{m_j}\}$$

$\mathcal{T}(\mathcal{S})$ – the set of all relevant dates for the forward swaps from a family $\mathcal{S}$, that is,

$$\mathcal{T}(\mathcal{S}) = \bigcup_{j=1}^{l} \mathcal{T}(S_j).$$

It is clear that a date T_i does not belong to $\mathcal{T}(\mathcal{S})$ whenever the corresponding column of the matrix C^b has all entries equal to zero. Manifestly, the non-uniqueness of a solution to the inverse problem holds in that case, since the deflated bond price $B^b(t, T_i)$ cannot be retrieved. This is indeed trivial since the variable x_i does not appear in any equation in the linear system $C^b \overline{x}^b = \overline{e}^b$. In what follows, we only consider families $\mathcal{S}$ that generate the tenor structure $\mathcal{T}$, in the sense that $\mathcal{T} = \mathcal{T}(\mathcal{S})$.

7.2.2 Graph Theory Terminology

Before proceeding to the detailed analysis of the inverse problem, we need first to recall some basic graph theory terminology, which will be employed in what follows. We also provide some preliminary results concerning the existence of a cycle in a graph, which is formally associated with a family of forward swap rates. For an in-depth introduction to the graph theory, the interested reader may consult the monograph by Bollobás [17].

Definition 7.2.3. A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of unordered pairs of V. The set V is called the *vertex set* and E is called the *edge set.*

Definition 7.2.4. (i) Two vertices v_1 and v_2 are said to be *adjacent* if there exists an edge $e \in E$ such that e is the unordered pair $\{v_1, v_2\}$.
(ii) A *path* is a finite sequence of vertices such that each of the vertex is adjacent to next vertex in the sequence.
(iii) A *cycle* is a path such that the start vertex and end vertex are the same.
(iv) A graph G is said to be *connected* if there exists a path between any two vertices.
(v) A graph G is *acyclic* if no cycles exist.

The next result is borrowed from [17]; it is in fact produced by combining Theorem 4 on page 7 and Exercise 9 on page 23 therein.

Theorem 7.2.1. *The following statements are equivalent for a graph G with m vertices:*
(i) G is a minimal connected graph, that is, G is connected and if $\{x, y\} \in E$ then $G - \{x, y\}$ is disconnected,
(ii) G is a maximal acyclic graph, that is, G is acyclic and if x and y any non-adjacent vertices of G then the graph $G + \{x, y\}$ contain a cycle,
(iii) G is connected and has $m - 1$ edges,
(iv) G is acyclic and has $m - 1$ edges.

Let $\mathcal{S}$ be a family consisting of l swaps on the complete tenor structure $\mathcal{T} = \{T_0, T_1, \dots, T_n\}$. As previously introduced, the set $\mathcal{T}_0(\mathcal{S})$ is the set of all start or maturity dates of swaps in $\mathcal{S}$. To translate our set-up into graph theory terminology, we regard the set $\mathcal{T}_0(\mathcal{S})$ as a vertex set. A swap $S_j \in \mathcal{S}$ which start at T_{s_j} and matures at T_{m_j} is characterized by the pair of elements $\{T_{s_j}, T_{m_j}\}$. This characterization allows us to regard $\mathcal{S}$ as the edge set. Note that we only need to deal here with graphs that have finite number of vertices.

Definition 7.2.5. The *graph of* $\mathcal{S}$ is the vertex and edge pair given as $(\mathcal{T}_0(\mathcal{S}), \mathcal{S})$. It is clear that the graph of $\mathcal{S}$ has l edges and at most $n+1$ vertices.

As will be elaborated on in Section 7.2.4 (see, in particular, Remark 7.2.2), the (non-)existence of a cycle in the graph of $\mathcal{S}$ does not seem to be of primary interest in the study of $\mathcal{T}$-admissibility of a family $\mathcal{S}$. However, in order to re-examine Proposition 2.1 in Galluccio et al. [42], we will first present some basic properties of the graph of $\mathcal{S}$ directly related to the (non-)existence of a cycle in this graph. The first lemma in this vein is a straightforward consequence of Theorem 7.2.1.

Lemma 7.2.1. *Let $\mathcal{S} = \{S_1, \dots, S_l\}$ be any family of swaps with $\mathcal{T}(\mathcal{S}) = \mathcal{T}$. If $l > n$ then there exists a cycle in the graph of $\mathcal{S}$.*

The next auxiliary result deals with the acyclic case.

Lemma 7.2.2. *Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a family of n swaps with $\mathcal{T}(\mathcal{S}) = \mathcal{T}$. If there are no cycles in the graph of $\mathcal{S}$ then each date from the tenor structure $\mathcal{T}$ is either the start date or the maturity date of some swap S_j from $\mathcal{S}$, that is, then the equality $\mathcal{T}_0(\mathcal{S}) = \mathcal{T}$ holds.*

Proof. Let us assume that there exists a date $T_k \in \mathcal{T}$ which does not belong to $\mathcal{T}_0(\mathcal{S})$. Then there exists a family $\widetilde{\mathcal{S}}$ comprising n swaps and such that $\mathcal{T}_0(\widetilde{\mathcal{S}}) = \{T_0 < \dots < T_{k-1} < T_{k+1} < \dots < T_n\}$. Indeed, to obtain $\widetilde{\mathcal{S}}$ from $\mathcal{S}$, it suffices to delete the date T_k from all swap equations. Hence, by Lemma 7.2.1, a cycle necessarily exists in the family $\widetilde{\mathcal{S}}$ and thus also in $\mathcal{S}$. □

The last auxiliary result furnishes a simple, but useful, observation.

Lemma 7.2.3. *Assume that $l = n$ and the graph $\mathcal{S}$ is connected. Then $\mathcal{T}_0(\mathcal{S}) = \mathcal{T}$ and thus there are no cycles in $\mathcal{S}$.*

7.2.3 Inverse Problem for Deflated Bonds

We are in a position to formalize the inverse problem for the deflated bond prices and the related concepts of admissibility of a family $\mathcal{S}$. Note that the term 'almost every' refers throughout to the Lebesgue measure on the respective space $\mathbb{R}^l$ (or $\mathbb{R}^l_+$).

Inverse Problem (IP.1) A family $\mathcal{S} = \{S_1, \dots, S_l\}$ admits a solution to the *inverse problem* (IP.1) if for any $b \in \{0, \dots, n\}$ and for almost every $(\kappa_1, \dots, \kappa_l) \in \mathbb{R}^l$ there exists a solution $\overline{x}^b \in \mathbb{R}^n$ to the linear system $C^b(\kappa_1, \dots, \kappa_l)\overline{x}^b = \overline{e}^b(\kappa_1, \dots, \kappa_l)$ associated with $\mathcal{S}$.

Galluccio et al. [42] prefer to deal directly with the stochastic linear system $\mathcal{C}^b\mathcal{B}^b = \overline{e}^b$ where $\mathcal{C}^b$ is the matrix of stochastic processes obtained by replacing the variables $(\kappa_1, \dots, \kappa_l)$ by diffusion-type forward swap rate processes $\kappa^j = \kappa^{s_j, m_j}$, $j = 1, \dots, l$ and the unknowns $(x_0, \dots, x_{b-1}, x_{b+1}, \dots, x_n) \in \mathbb{R}^n$ are replaced by the (unknown) deflated bond price processes $B^b(t, T_i)$, $i = 0, \dots, b-1, b+1, \dots, n$. To the best of our understanding, they work under the tacit assumption that for any $t \in (0, T_0]$ the random variable $(\kappa^1_t, \dots, \kappa^l_t)$ has a strictly positive joint probability density function on $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$, so that the probability distribution of $(\kappa^1_t, \dots, \kappa^l_t)$ has the full support $\mathbb{R}^l$. In these circumstances, the inverse problem (IP.1) introduced above appears to be equivalent to the inverse problem studied in [42].

The following definition reflects the idea of admissibility of a family of forward swaps, as proposed by Galluccio et al. [42]. It is based on the inverse problem (IP.1), but it also refers to some additional features of a solution to this problem, such as its uniqueness and positivity.

Definition 7.2.6. We say that a family $\mathcal{S}$ of forward swaps associated with $\mathcal{T}$ is *weakly $\mathcal{T}$-admissible* if for any choice of $b \in \{0, \dots, n\}$ the following property holds: for almost every $(\kappa_1, \dots, \kappa_l) \in \mathbb{R}^l$ there exists a unique non-zero solution $\overline{x}^b \in \mathbb{R}^n$ to the linear system $C^b(\kappa_1, \dots, \kappa_l)\overline{x}^b = \overline{e}^b(\kappa_1, \dots, \kappa_l)$ associated with $\mathcal{S}$. If, in addition, the solution is strictly positive for almost every $(\kappa_1, \dots, \kappa_l) \in \mathbb{R}^l_+$ then we say that a family $\mathcal{S}$ is *$\mathcal{T}$-admissible.*

We write $\mathfrak{A}$ to denote the class of all families of forward swaps that are weakly $\mathcal{T}$-admissible, whereas $\overline{\mathfrak{A}}$ stands for the class of all families that are $\mathcal{T}$-admissible.

Remark 7.2.1. The property of $\mathcal{T}$-admissibility is the strongest form of invertibility, which appears to be the most convenient from the viewpoint of further applications. If it holds then the Radon-Nikodým densities for various swap measures can be computed from the family $\mathcal{B}^b$ and expressed in terms of the underlying forward swap rates. Consequently, in that case it will be easy to derive the joint dynamics of forward swap rates under a single probability measure. In addition, these dynamics will be supported by an arbitrage-free model of positive deflated bond prices.

Before proceeding, let us comment on the approach presented in the paper by Galluccio et al. [42]. In [42], the authors formalize the concept of admissibility of a family $\mathcal{S}$ through the following definition that corresponds to Definition 2.2 in [42]. For an overview of the relevant terminology from graph theory, we refer to Section 7.2.2.

Definition 7.2.7. A family $\mathcal{S}$ of forward swaps associated with $\mathcal{T}$ is *admissible* if the following conditions are satisfied:
(i) the number of forward swaps in $\mathcal{S}$ equals n, that is, $l = n$,
(ii) any date $T_i \in \mathcal{T}$ coincides with the reset/settlement date of at least one forward swap from $\mathcal{S}$,
(iii) there are no cycles (also called *degenerate subsets* in [42]) in $\mathcal{S}$.

The main theoretical result in [42] states that the admissibility of $\mathcal{S}$, in the sense of Definition 7.2.7, is a necessary and sufficient condition for the following property: a set of non-zero deflated bond prices associated with $\mathcal{S}$ exists and is unique, $\mathbb{P}$-a.s., for any choice of the bond numéraire and any generic process $(\kappa^1, \dots, \kappa^l)$ for forward swap rates.

In other words, the main result in [42] (see Proposition 2.1 therein) establishes the equivalence between the admissibility of $\mathcal{S}$ and the existence of a unique solution to their inverse problem, $\mathbb{P}$-a.s.

As we shall argue in what follows, however, this result is not true, in general. As a consequence, we observe that Definition 7.2.7 of admissibility of a family $\mathcal{S}$ could be misleading, since it implicitly hinges on the validity of Proposition 2.1 in [42]. It is our understanding that the method of proof of Proposition 2.1 in [42] hinges on the following conjectures:

- If the number l of forward swaps in $\mathcal{S}$ differs from n then either there is no solution to the linear system $C^b\overline{x}^b = \overline{e}^b$ or the non-uniqueness of a solution holds. Therefore, the necessary requirement is that C^b should be a square matrix, so that $l = n$.
- If $l = n$, but a *cycle* exists in $\mathcal{S}$ (see Section 7.2.2 below), then the rows in the matrix C^b are linearly dependent, and thus the rank of C^b is less than n, at least for a certain choice of $b \in \{0, \dots, n\}$. Consequently, there is no guarantee that a solution to the linear system $C^b\overline{x}^b = \overline{e}^b$ exists, for almost all realizations of forward swap rates and for an arbitrary choice of $B(t, T_b)$ as a numéraire bond.
- Otherwise, that is, when $l = n$ and there are no cycles in $\mathcal{S}$ then, for any choice of $b \in \{0, \dots, n\}$, the rows in the matrix C^b are linearly independent and thus the matrix C^b is non-singular, for almost all realizations of forward swap rates. Consequently, the unique solution to $C^b\overline{x}^b = \overline{e}^b$ exists with probability one for any diffusion-type dynamics of forward swap rates. Moreover,

for almost all realizations of forward swap rates, the solution $\overline{x}^b$ is non-zero, meaning that $x_i \neq 0$ for every $i = 0, \ldots, n$ (recall that $x_b = 1$).

We note that the proof of Proposition 2.1 in [42] suffers from the following shortcomings:

- First, the proof of the sufficiency clause is based on the argument that if the first column has only one non-zero entry then the row vectors are linearly independent. This is, of course, not true in general (it seems that the authors assume that the coefficients in the linear combination should all be non-zero, whereas in fact it suffices that not all of them vanish). A similar argument is used when there a several non-zero entries.

- Second, the proof of the necessity clause is based on the observation that the sum of equations corresponding to the upper and lower paths in a cycle yields an equation of a special shape. This fact does not prove, however, that the row vectors are linearly dependent, as was claimed in [42]. To illustrate this point, we present below an example of a family $\mathcal{S}$ with a cycle for which for any choice of $b \in \{0, \ldots, n\}$ the corresponding matrix C^b is non-singular, for almost all $(\kappa_1, \ldots, \kappa_l) \in \mathbb{R}^l$.

Example 7.2.1. The following simple counter-example shows that a family $\mathcal{S}$ with a cycle can be weakly $\mathcal{T}$-admissible. Let $n = 3$ and let (s_j, m_j), $j = 1, 2, 3$ be given as $(0, 2), (2, 3)$ and $(0, 3)$, respectively. The swaps S_1 and S_2 yield the path (T_0, T_3) and this path is also given by the swap S_3, so that a cycle (T_0, T_0) is present. For $b = 0$, we obtain the following linear system

$$C^0 \overline{x}^0 = \begin{bmatrix} c_{1,1} & \widetilde{c}_{1,2} & 0 \\ 0 & -1 & \widetilde{c}_{2,3} \\ c_{3,1} & c_{3,2} & \widetilde{c}_{3,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \overline{e}^0.$$

One can check by direct computations that, for any choice of $b \in \{0, 1, 2, 3\}$, the following properties hold: (i) for almost all $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3$, the matrix C^b is non-singular and (ii) for almost all $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3$, the unique solution $\overline{x}^b$ has all entries non-zero. We conclude that the considered family $\mathcal{S}$ is not admissible, in the sense of Definition 7.2.7, but it satisfies the Definition 7.2.6 of the weak $\mathcal{T}$-admissibility. This example makes it clear that the necessity clause in Proposition 2.1 in [42] is incorrect. We will later comment on the sufficiency clause in Proposition 2.1 in that paper.

Let us conclude this subsection by mentioning that although Galluccio et al. [42] examine the admissibility of a family of forward swaps with respect the tenor structure $\mathcal{T}$ in the context of the construction of swap market models, they remain silent on the joint dynamics of forward swaps, and they focus instead on the dynamics of each forward swap rate under the corresponding swap martingale measure. However, one can infer from the discussion in Section 2 in [42] that when a family $\mathcal{S}$ is admissible then one can specify the model by choosing arbitrarily the volatilities in formula (2.2) in [42]. The implicit goal is to obtain a generic arbitrage-free model supported by the unique family of non-zero deflated bond prices. We will return to this issue in Section 7.3.2.

7.2.4 Weak Admissibility of Forward Swaps

The purpose of this subsection is to provide necessary and sufficient conditions for a family of forward swaps to be weakly $\mathcal{T}$-admissible. We postulate throughout that a family $\mathcal{S}$ of forward swaps is such that the equality $\mathcal{T}(\mathcal{S}) = \mathcal{T}$ holds. Equivalently, for an arbitrary choice of $b \in \{0, \ldots, n\}$ there is no null column in the matrix C^b associated with $\mathcal{S}$.

Preliminary Results

Before examining various forms of admissibility, let us quickly note that the following property holds.

Lemma 7.2.4. *Let us fix $(\kappa_1,\ldots,\kappa_l)\in\mathbb{R}^l$. The following properties are equivalent:*
(i) the system $C^b(\kappa_1,\ldots,\kappa_l)\overline{x}^b=\overline{e}^b(\kappa_1,\ldots,\kappa_l)$ has a non-zero solution for some $b\in\{0,\ldots n\}$,
(ii) the system $C^b(\kappa_1,\ldots,\kappa_l)\overline{x}^b=\overline{e}^b(\kappa_1,\ldots,\kappa_l)$ has a non-zero solution for all $b\in\{0,\ldots n\}$.

Proof. (i) $\Rightarrow$ (ii) If a non-zero solution exists for a particular b, then for any $\widetilde{b}\in\{0,\ldots n\}$ the solution to the system $C^{\widetilde{b}}\overline{x}^{\widetilde{b}}=\overline{e}^{\widetilde{b}}$ can be obtained by taking the ratios of the solution to the system $C^b\overline{x}^b=\overline{e}^b$. This argument corresponds to the simple observation that to obtain the deflated bond prices for any other choice of a numéraire bond, one may use the equality

$$B^{\widetilde{b}}(t,T_i)=\frac{B^b(t,T_i)}{B^b(t,T_{\widetilde{b}})},\quad\forall\,\widetilde{b}\neq b.$$

The implication (ii) $\Rightarrow$ (i) is obvious. □

As we have already seen from Example 7.2.1, the non-existence of a cycle is not a necessary condition for the weak $\mathcal{T}$-admissibility of a family of swaps. We therefore introduce the concept of $(\mathcal{T},b)$-inadmissibility, which will prove useful in the sequel.

Definition 7.2.8. Let us fix $b\in\{0,\ldots,n\}$. A subset $\widetilde{\mathcal{S}}$ of a family $\mathcal{S}$ of swaps is said to be $(\mathcal{T},b)$-*inadmissible* if the number of swaps in $\widetilde{\mathcal{S}}$ is strictly greater than the number of dates in $\mathcal{T}(\widetilde{\mathcal{S}})\setminus\{T_b\}$. We denote by $\mathfrak{M}_0$ the class of all families of swaps that do not contain a $(\mathcal{T},b)$-inadmissible subset for every $b\in\{0,\ldots,n\}$.

Note that the number of dates in $\mathcal{T}(\widetilde{\mathcal{S}})\setminus\{T_b\}$ is equal to the number of all variables from the set $x_0,\ldots,x_{b-1},x_{b+1},\ldots,x_n$ that are associated with a given subset $\widetilde{\mathcal{S}}$.

The following properties can be checked by inspection:
(a) if a subset $\widetilde{\mathcal{S}}$ is $(\mathcal{T},b)$-inadmissible for some b such that $T_b\in\mathcal{T}(\widetilde{\mathcal{S}})$ then it is also $(\mathcal{T},\widetilde{b})$-inadmissible for any $\widetilde{b}$ such that $T_{\widetilde{b}}\in\mathcal{T}(\widetilde{\mathcal{S}})$.
(b) if a subset $\widetilde{\mathcal{S}}$ is $(\mathcal{T},b)$-inadmissible for some b such that $T_b\notin\mathcal{T}(\widetilde{\mathcal{S}})$ then it is also $(\mathcal{T},\widetilde{b})$-inadmissible for any $\widetilde{b}\in\{0,\ldots,n\}$.

Example 7.2.2. The family $\mathcal{S}$ introduced in Example 7.2.1 represents a cycle, denoted as $\mathcal{S}_c$, which is not a $(\mathcal{T},b)$-inadmissible family for any choice of $b\in\{0,\ldots,3\}$. Indeed, we have here $\mathcal{S}_c=\mathcal{S}$ and $\mathcal{T}(\mathcal{S}_c)=\mathcal{T}(\mathcal{S})=\mathcal{T}$. Hence, for any $b\in\{0,\ldots,3\}$, the number of dates in the set $\mathcal{T}(\mathcal{S}_c)\setminus\{T_b\}$ is equal to 3 and thus it always coincides with the number of forward swaps in $\mathcal{S}_c$.

Remark 7.2.2. The existence a $(\mathcal{T},b)$-inadmissible cycle $\mathcal{S}_c$ for some $b\in\{0,\ldots,n\}$ implies the existence of a $(\mathcal{T},\widetilde{b})$-inadmissible subset $\widetilde{\mathcal{S}}$ for some $\widetilde{b}\in\{0,\ldots,n\}$ (simply take $b=\widetilde{b}$). Note, however, that the existence of a $(\mathcal{T},\widetilde{b})$-inadmissible subset $\widetilde{\mathcal{S}}$ for some $\widetilde{b}\in\{0,\ldots,n\}$ does not imply the existence of a $(\mathcal{T},b)$-inadmissible cycle $\mathcal{S}_c$, in general. This feature can be illustrated through the following counter-example.

Example 7.2.3. The forward swaps that we will consider here are standard forward swaps, so that all dates between the starting date and maturity date are settlement dates. Let $\widetilde{\mathcal{S}}$ be equal to $\mathcal{S}_c^1\cup\mathcal{S}_c^2$, where

$$\mathcal{S}_c^1=\{S_0=\{T_0,T_3\},S_1=\{T_0,T_4\},S_2=\{T_3,T_4\}\}$$

and $S_c^2=\{S_3=\{T_1,T_2\},S_4=\{T_2,T_5\},S_5=\{T_1,T_5\}\}$. It is easy to see that $\widetilde{\mathcal{S}}$ is a $(\mathcal{T},b)$-inadmissible subset for every $b=\{0,\ldots,5\}$. However, the only cycles $\mathcal{S}_c^1$ and $\mathcal{S}_c^2$ in the family $\widetilde{\mathcal{S}}$ are clearly not $(\mathcal{T},b)$-inadmissible cycles for any $b=\{0,\ldots,5\}$.

We denote by $\mathfrak{C}$ ($\mathfrak{N}_c$, resp.) the class of all families of forward swaps that contain (do not contain, resp.) a cycle.

Proposition 7.2.1. *The existence of a $(\mathcal{T},b)$-inadmissible subset for some b implies the existence of a cycle. In particular, the strict inclusion $\mathfrak{N}_c\subsetneq\mathfrak{M}_0$ holds.*

Proof. By definition, there exists a subset of swaps $\widetilde{\mathcal{S}}$ such that the number of relevant dates (variables) is strictly less than the number of swaps in the $\widetilde{\mathcal{S}}$. Now, let us consider the graph $(\mathcal{T}_0(\widetilde{\mathcal{S}}), \widetilde{\mathcal{S}})$. Since $\widetilde{\mathcal{S}}$ is a $(\mathcal{T}, b)$-inadmissible subset, the number of edges is strictly greater than the number of vertices. By Theorem 7.2.1, there must exist a cycle (any other relevant date that is not in $\mathcal{T}(\widetilde{\mathcal{S}})$, but falls between the smallest and largest date in $\mathcal{T}(\widetilde{\mathcal{S}})$, manifestly stretches a cycle). We conclude that the inclusion $\mathfrak{N}_c \subset \mathfrak{M}_0$ holds. However, from Example 7.2.1, we deduce that the inclusion is in fact strict. □

In view of Proposition 7.2.1, the condition of the non-existence of a cycle in $\mathcal{S}$ is stronger than the assumption of the non-existence of a $(\mathcal{T}, b)$-inadmissible subset, which will be shown to be a necessary condition for the weak $\mathcal{T}$-admissibility (see Proposition 7.2.2). Recall also that Example 7.2.1 describes a weakly $\mathcal{T}$-admissible family $\mathcal{S}$ that contains a cycle. We thus see that the non-existence of a cycle is not a necessary condition for the weak $\mathcal{T}$-admissibility, that is, the class $\mathfrak{A}$ is not included in $\mathfrak{N}_c$.

The question whether the non-existence of a cycle is a sufficient condition for the weak $\mathcal{T}$-admissibility, that is, whether $\mathfrak{N}_c \subsetneq \mathfrak{A}$ (note that Example 7.2.1 shows that $\mathfrak{N}_c \neq \mathfrak{A}$) remains open (see Question 7.2.1 below).

Necessary Conditions for the Weak $\mathcal{T}$-Admissibility

Our goal is to show that the property that a family $\mathcal{S}$ belongs to the class $\mathfrak{M}_0$ is a necessary condition for the weak $\mathcal{T}$-admissibility of a family $\mathcal{S}$. Unfortunately, it appears that this condition is not sufficient for the weak $\mathcal{T}$-admissibility and thus further studies will be needed.

We continue the study of the class $\mathfrak{M}_0$. By a *vertical block* in a matrix, we mean any collection of its rows in which the number of non-null columns is less than the number of rows. According to the standard terminology of linear algebra, a 'vertical block' represents an *overdetermined subsystem*. However, for brevity, we prefer to use a more intuitive term vertical block. By a *permutation* of the matrix C^b we mean a finite number of either row or column swaps (or both) performed on the matrix C^b.

Lemma 7.2.5. *Let us fix some $b \in \{0, \ldots, n\}$. The following properties are equivalent:*
(i) there exists a $(\mathcal{T}, b)$-inadmissible subset $\widetilde{\mathcal{S}}$,
(ii) there exists a vertical block in the matrix C^b after some permutation.

Proof. It is clear that any $(\mathcal{T}, b)$-inadmissible subset $\mathcal{S}$ corresponds, after a suitable permutation, to a vertical block, so that (i) implies (ii). The converse follows immediately from the definition of a $(\mathcal{T}, b)$-inadmissible subset. □

The following result shows that the existence of a $(\mathcal{T}, b)$-inadmissible subset in $\mathcal{S}$ implies the non-existence of a unique solution to $C^b\overline{x}^b = \overline{e}^b$, at least for a certain $b \in \{0, \ldots, n\}$.

Proposition 7.2.2. *If a family $\mathcal{S}$ contains a $(\mathcal{T}, b)$-inadmissible subset $\widetilde{\mathcal{S}}$ for some $b \in \{0, \ldots, n\}$ then $\mathcal{S}$ is not weakly $\mathcal{T}$-admissible. Consequently, the inclusion $\mathfrak{A} \subset \mathfrak{M}_0$ is valid.*

Proof. It suffices to focus on the variables corresponding to the dates from the set $\mathcal{T}(\widetilde{\mathcal{S}}) \setminus T_b$, where b is such that the subset $\widetilde{\mathcal{S}}$ is $(\mathcal{T}, b)$-inadmissible.

If $T_b \notin \mathcal{T}(\widetilde{\mathcal{S}})$ then these variables need to satisfy an overdetermined homogeneous linear system. We then deal with the following two subcases: (i) if the determinant of a square subsystem is a non-zero rational function then the null solution is the unique solution to the particular subset of unknowns, (ii) the determinant of a square subsystem is zero and thus the solution either fails to exist or is not unique. Therefore, we see that in both cases the family $\mathcal{S}$ is not weakly $\mathcal{T}$-admissible.

If, on the contrary, $T_b \in \mathcal{T}(\widetilde{\mathcal{S}})$ then these variables need to satisfy a non-homogeneous linear system in which the number of equations is larger than the number of variables. Then by changing the date T_b to some date $T_{\widetilde{b}} \notin \mathcal{T}(\widetilde{\mathcal{S}})$, we obtain a once again a homogeneous linear system, in which the number of equations is at least equal to the number of variables. Therefore, we are back in either case (i) or case (ii) (note that if all dates are already in $\mathcal{T}(\widetilde{\mathcal{S}})$, that is, when $\mathcal{T} = \mathcal{T}(\widetilde{\mathcal{S}})$, we can add one more date to $\mathcal{T}$ in order to perform this step). □

It is important to observe that the inclusion $\mathfrak{A} \subset \mathfrak{M}_0$ is in fact strict, that is, $\mathfrak{A} \subsetneq \mathfrak{M}_0$, in general. Specifically, for any $n \geq 2$ there exists a family $\mathcal{S}$ of forward swaps such that $\mathcal{S} \in \mathfrak{A}^c \cap \mathfrak{M}_0$ (e.g., it is fairly easy to produce an example of a family with a cycle, which is not weakly $\mathcal{T}$-admissible). We thus conclude that the property that a family $\mathcal{S}$ belongs to the class $\mathfrak{M}_0$ is a necessary, but not sufficient, condition for the weak $\mathcal{T}$-admissibility of a family $\mathcal{S}$.

As a preliminary step towards identification of a sufficient condition for the weak $\mathcal{T}$-admissibility of a family $\mathcal{S}$, we establish a simple result furnishing a stronger necessary condition.

Lemma 7.2.6. *If for some $b \in \{0, \ldots, n\}$, the linear system $C^b \overline{x}^b = \overline{e}^b$ associated with $\mathcal{S}$ has a vertical block after a finite number of row operations then the family of forward swaps $\mathcal{S}$ is not weakly $\mathcal{T}$-admissible.*

Proof. Suppose that after a finite number of row operations there exists a vertical block and either:
(i) the vertical block forms a homogeneous overdetermined subsystem, or
(ii) the vertical block forms a non-homogeneous overdetermined subsystem.

In case (i), if any square subsystem has non-zero determinant then the null solution is the unique solution to the subsystem. If, on the contrary, any square subsystem has zero determinant then the original matrix is not invertible.

In case (ii), one can adjust the choice of a numéraire bond. By picking $B(t, T_{\widetilde{b}})$ such that x_b is not in the overdetermined subsystem, the non-homogeneous overdetermined subsystem becomes a homogeneous overdetermined subsystem and we are back in case (i) (similar argument applies if the subsystem is square) and we conclude that the family $\mathcal{S}$ is not weakly $\mathcal{T}$-admissible. □

An $l \times n$ matrix ($l \leq n$) is said to be *diagonalizable* if there exists a permutation such that the diagonal entries of an $l \times l$ sub-matrix are non-zero. In our set-up, we may also formulate the following definition.

Definition 7.2.9. For a fixed $b \in \{0, \ldots, n\}$, we say that the matrix C^b is *diagonalizable* if there exists a permutation of C^b such that all entries on the first diagonal of C^b are non-zero. We denote by $\mathfrak{D}$ the class of all families of forward swaps such that the matrix C^b is diagonalizable, for all $b \in \{0, \ldots, n\}$.

The proof of the following result is deferred to the appendix.

Proposition 7.2.3. *The following properties are equivalent, for any fixed $b \in \{0, \ldots, n\}$:*
(i) the matrix C^b is diagonalizable,
(ii) there is no $(\mathcal{T}, b)$-inadmissible subset $\widetilde{\mathcal{S}}$ in $\mathcal{S}$.
Consequently, the equality $\mathfrak{D} = \mathfrak{M}_0$ holds.

Recall that, by Lemma 7.2.5, the existence of a vertical block in the matrix C^b is equivalent to the existence of a $(\mathcal{T}, b)$-inadmissible subset $\widetilde{\mathcal{S}}$. Let us then suppose that for some b there exists a vertical block in C^b after some permutation of variables. It is worth noting that then it may still be true that the matrix C^b can by diagonalized for another choice of b.

Example 7.2.4. Unfortunately, it may also happen that the matrix C^b is diagonalizable for every $b \in \{0, \ldots, n\}$, but it fails to have a non-zero determinant for some $b \in \{0, \ldots, n\}$. To illustrate this claim, let us consider the following family of standard forward swaps:

$$\mathcal{S} = \big\{S_1 = \{T_0, T_4\}, S_2 = \{T_0, T_5\}, S_3 = \{T_0, T_6\}, S_4 = \{T_4, T_5\}, S_5 = \{T_5, T_6\}, S_6 = \{T_1, T_3\}\big\}.$$

Then, for every $b \in \{0,1,2,3,4,5,6\}$, the matrix C^b associated with the family $\mathcal{S}$ can be checked to be diagonalizable. However, the determinant of the matrix C^6 vanishes identically, since we have that

$$C^6 = \begin{bmatrix} -1 & a_1\kappa_1 & a_2\kappa_1 & a_3\kappa_1 & 1+a_4\kappa_1 & 0 \\ -1 & a_1\kappa_2 & a_2\kappa_2 & a_3\kappa_2 & a_4\kappa_2 & 1+a_5\kappa_2 \\ -1 & a_1\kappa_3 & a_2\kappa_3 & a_3\kappa_3 & a_4\kappa_3 & a_5\kappa_3 \\ 0 & 0 & 0 & 0 & -1 & 1+a_5\kappa_4 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & a_2\kappa_6 & 1+a_3\kappa_6 & 0 & 0 \end{bmatrix}, \qquad e^6 = \begin{bmatrix} 0 \\ 0 \\ 1+a_6\kappa_3 \\ 0 \\ 1+a_6\kappa_5 \\ 0 \end{bmatrix}.$$

It is easy to see that the fourth column can be written as linear combinations of second and third columns. This implies, of course, that the determinant of C^6 vanishes identically, and thus we conclude that the diagonalizability of the matrix C^b for every b is not a sufficient condition for the weak $\mathcal{T}$-admissibility of a family of forward swaps. Put another way, the property that $\mathcal{S}$ belongs to $\mathfrak{D}$ does not imply that $\mathcal{S}$ is in $\mathfrak{A}$. Since we already know that $\mathfrak{A} \subset \mathfrak{M}_0$ (see Proposition 7.2.2) and $\mathfrak{D} = \mathfrak{M}_0$ (from Proposition 7.2.3), we conclude that $\mathfrak{A} \subsetneq \mathfrak{M}_0$.

In what follows, by an *internal date* of a forward swap we mean a relevant date that is neither the start nor the maturity date in a given swap.

Definition 7.2.10. A family of forward swaps $\mathcal{S}$ is said to have an *inadmissible subset* if it has the following characteristics: there exist k dates for some $k \geq 2$ such that (i) they are internal dates to at least two swaps and (ii) they are relevant to at most $k-2$ other swaps. We denote by $\mathfrak{M}_1$ the class of all families of forward swaps that do not contain an inadmissible subset.

The next result furnishes a necessary condition for the weak $\mathcal{T}$-admissibility of a family $\mathcal{S}$.

Proposition 7.2.4. *The non-existence of inadmissible subsets is necessary for a family of swaps to be weakly $\mathcal{T}$-admissible, so that $\mathfrak{A} \subseteq \mathfrak{M}_1$.*

Proof. Suppose a family of forward swaps $\mathcal{S}$ has an inadmissible subset. To show that $\mathcal{S}$ is not weakly $\mathcal{T}$-admissible, we pick b outside of those k internal dates and concentrate on those k internal dates (as usual, represented by the columns in C^b). Without loss of generality, we may pick one column as the pivot column and perform column operations to eliminated other columns. As a consequence, we will end up with the subsystem without the pivot column, which consists of $k-1$ columns and at most $k-2$ non-zero rows. This gives the linear dependence of the columns. The following example will make this argument more explicit; a moment's reflection will show the generality of our reasoning. To illustrate the method, we focus on columns number two, three and four in the matrix in Example 7.2.4

$$\begin{bmatrix} a_1\kappa_1 & a_2\kappa_1 & a_3\kappa_1 \\ a_1\kappa_2 & a_2\kappa_2 & a_3\kappa_2 \\ a_1\kappa_3 & a_2\kappa_3 & a_3\kappa_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a_2\kappa_6 & 1+a_3\kappa_6 \end{bmatrix} \xrightarrow[c_2=\frac{a_2}{a_1}c_1-c_2]{c_3=\frac{a_3}{a_1}c_1-c_3} \begin{bmatrix} a_1\kappa_1 & 0 & 0 \\ a_1\kappa_2 & 0 & 0 \\ a_1\kappa_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a_2\kappa_6 - \frac{a_2}{a_1} & 1+a_3\kappa_6 - \frac{a_3}{a_1} \end{bmatrix}.$$

It is now clear that the columns are linearly dependent. By definition, this implies this family of swap is not weakly $\mathcal{T}$-admissible. ☐

Proposition 7.2.5. *The existence of an inadmissible subset implies the existence of a cycle, so that $\mathfrak{M}_1^c \subset \mathfrak{C}$. Moreover, the strict inclusion $\mathfrak{N}_c \subsetneq \mathfrak{M}_1$ holds, in general.*

Proof. By definition of the class $\mathfrak{M}_1^c$, there exist k internal dates such that at most $k-2$ swaps either start or mature on those k internal dates. Therefore, there must be at least $n-k+2$ swaps starting and maturing outside the k internal dates. If one consider the graph of $\mathcal{S}$ without those k

internal dates, it is a graph with $n-k$ vertices and at least $n-k+2$ edges and thus, by Theorem 7.2.1, a cycle necessarily exists in $\mathcal{S}$. Hence, the inclusion $\mathfrak{M}_1^c \subset \mathfrak{C}$ is valid. For the second inclusion, we also observe that Proposition 7.2.4 yields $\mathfrak{A} \subseteq \mathfrak{M}_1$, whereas Example 7.2.1 describes a family $\mathcal{S}$ with $n=3$ belonging to $\mathfrak{A} \cap \mathfrak{C}$. This also means that $\mathfrak{M}_1^c \subsetneq \mathfrak{C}$, in general. □

Sufficient Conditions for the Weak $\mathcal{T}$-Admissibility

Throughout this section, we set $l=n$ and we consider an arbitrary family $\mathcal{S}=\{S_1,\dots,S_n\}$ of forward swaps. Our next goal is to provide a sufficient condition for the weak $\mathcal{T}$-admissibility of a family $\mathcal{S}$, that is, the existence and uniqueness of a non-zero solution to the linear system $C^b\overline{x}^b=\overline{e}^b$, for an arbitrary choice of $b\in\{0,\dots,n\}$, for almost every $(\kappa_1,\dots,\kappa_n)\in\mathbb{R}^n$.

Definition 7.2.11. We denote by $\overline{\mathfrak{D}}$ the class of all diagonalizable families $\mathcal{S}$ of n forward swaps such that the diagonalizability property of the matrix C^b is preserved under elementary row operations, for all $b\in\{0,\dots,n\}$.

Remark 7.2.3. From Example 7.2.4, we see that the diagonalizability of the matrix C^b is not preserved under column operations. Since the row rank and column rank of any matrix are the same, for simplicity, we decided to work with row operations in the above definition.

It is fair to acknowledge that although the class $\overline{\mathfrak{D}}$ is suitable to give a necessary and sufficient condition for the weak $\mathcal{T}$-admissibility, it may be difficult to check whether a given family $\mathcal{S}$ does indeed belong to the class $\overline{\mathfrak{D}}$ or not.

Proposition 7.2.6. *(i) If a family $\mathcal{S}$ belongs to $\overline{\mathfrak{D}}$ then $\mathcal{S}$ is weakly $\mathcal{T}$-admissible and thus the inclusion $\overline{\mathfrak{D}}\subset\mathfrak{A}$ is valid.*
(ii) The converse inclusion $\mathfrak{A}\subset\overline{\mathfrak{D}}$ holds as well.
Consequently, the equality $\mathfrak{A}=\overline{\mathfrak{D}}$ is true.

Proof. We first prove (i), that is, we establish a sufficient condition for the weak $\mathcal{T}$-admissibility of $\mathcal{S}$. By assumption, a family $\mathcal{S}$ is in $\mathfrak{D}$ and thus the matrix C^b is diagonalizable for every $b\in\{0,\dots,n\}$. Therefore, for a fixed $b\in\{0,\dots,n\}$, the matrix C^b can be diagonalized. Next, we take the first row as the pivot row and we eliminate the entries below the first diagonal and, by the assumption that $\mathcal{S}$ belongs to $\overline{\mathfrak{D}}$, the sub-matrix is again diagonalizable. The procedure then continues with the second row becoming the pivot row, so that the entries below the second diagonal are eliminated. At each stage, the assumption that diagonalizability is preserved is used to diagonalize the matrix after each row operation. After completing this procedure, we obtain a matrix in row echelon form with the diagonals consisting of non-zero rational functions in variables $\kappa_1,\dots,\kappa_n$. It is then easy to see that the determinant is non-zero, for almost all generic values of $(\kappa_1,\dots,\kappa_n)\in\mathbb{R}^n$. This ends the proof of the inclusion $\overline{\mathfrak{D}}\subset\mathfrak{A}$.

For part (ii), we need to show that $\mathfrak{A}\subset\overline{\mathfrak{D}}$. We already know from Propositions 7.2.2 and 7.2.3 that the inclusion $\mathfrak{A}\subset\mathfrak{D}$ holds. It thus suffices to show that $\overline{\mathfrak{D}}^{\,c}\cap\mathfrak{A}=\emptyset$. To this end, we will argue by contradiction. Let us then assume that the diagonalizability property is not preserved under elementary row operations for all $b\in\{0,\dots,n\}$, but the linear system $C^b\overline{x}^b=\overline{e}^b$ associated with $\mathcal{S}$ has a unique non-zero solution for any choice of b. The first condition implies that for some $\widetilde{b}\in\{0,\dots,n\}$, the matrix $C^{\widetilde{b}}$ is no longer diagonalizable (that is, there exists a vertical block) after a finite number of elementary row operations. Using Lemma 7.2.6, we conclude that the family $\mathcal{S}$ is not weakly $\mathcal{T}$-admissible. However, we also know that the solution of the linear system is invariant under elementary row operations and thus we arrive at the contradiction. □

Remark 7.2.4. We emphasize once again that it is important to assume that, for all $b\in\{0,\dots,n\}$, the matrix C^b belongs to $\overline{\mathfrak{D}}$, that is, the diagonalization property is invariant under elementary row operations. The weaker assumption that for all $b\in\{0,\dots,n\}$, the matrix C^b is diagonalizable is not sufficient for the weak $\mathcal{T}$-admissibility, as was illustrated through Example 7.2.4.

We will now summarize the results regarding the weak $\mathcal{T}$-admissibility of a family $\mathcal{S}$ obtained so far. For the reader's convenience, we recall the notation for the relevant classes of families $\mathcal{S}$ of forward swaps:
$\mathfrak{A}$ – weakly $\mathcal{T}$-admissible families;
$\mathfrak{C}$ – families that contain a cycle;
$\mathfrak{N}_c$ – families that do not contain a cycle;
$\mathfrak{D}$ – families such that the matrix C^b is diagonalizable, for all $b \in \{0, \dots, n\}$;
$\mathfrak{M}_0$ – families that do not contain a $(\mathcal{T}, b)$-inadmissible subset, for all $b \in \{0, \dots, n\}$;
$\mathfrak{M}_1$ – families that do not contain an inadmissible subset;
$\overline{\mathfrak{D}}$ – families from $\mathfrak{D}$ for which the diagonalization property of C^b is preserved under elementary row operations, for all $b \in \{0, \dots, n\}$.

Theorem 7.2.2. *Let $\mathcal{T} = \{T_0, \dots, T_n\}$ be a fixed tenor structure. We consider the class of all families $\mathcal{S} = \{S_1, \dots, S_n\}$ of forward swaps associated with $\mathcal{T}$. Then the following properties hold:*
(i) $\mathfrak{A} = \overline{\mathfrak{D}} \subsetneq \mathfrak{D} = \mathfrak{M}_0$;
(ii) $\mathfrak{A} \subset \mathfrak{D} \cap \mathfrak{M}_1 \subset \mathfrak{D} = \mathfrak{M}_0$;
(iii) $\mathfrak{N}_c \subsetneq \mathfrak{D} = \mathfrak{M}_0$ *and* $\mathfrak{N}_c \subsetneq \mathfrak{M}_1$;
(iv) $\mathfrak{A} \cap \mathfrak{C} \neq \emptyset$.

Proof. It suffices to put together the partial results established thus far:
(i) it suffices to combine Propositions 7.2.2, 7.2.3 and 7.2.6 with Example 7.2.4;
(ii) the first inclusion follows from Propositions 7.2.2 and 7.2.4; the second is obvious;
(iii) Propositions 7.2.1 and 7.2.3 yield the first part in (iii); the second part follows from Proposition 7.2.5;
(iv) Example 7.2.1 describes a family $\mathcal{S}$ with $n = 3$ belonging to $\mathfrak{A} \cap \mathfrak{C}$.

□

Recall that we have argued that the (non-)existence of a cycle is not of primary interest, as opposed to the (non-)existence of a $(\mathcal{T}, b)$-admissible subset. As already mentioned, when analyzing the necessity clause, Galluccio et al. [42] failed to realize that not every cycle is a $(\mathcal{T}, b)$-inadmissible subset. Unfortunately, the sufficiency clause in the main result in Galluccio et al. [42] remains still unclear to us, since we were unable to either establish the inclusion $\mathfrak{N}_c \subsetneq \mathfrak{A}$ or invalidate it by means of a counterexample. This motivates us to formulate the following problem which, to the best of our knowledge, remains open.

Question 7.2.1. Is the non-existence of a cycle in a family of forward swaps $\mathcal{S}$ a sufficient condition for $\mathcal{S}$ the weak $\mathcal{T}$-admissibility of $\mathcal{S}$, i.e., does the inclusion $\mathfrak{N}_c \subsetneq \mathfrak{A}$ hold true?

Alternatively, one may attempt to answer the following question.

Question 7.2.2. Is the diagonalizability property of a family of forward swaps $\mathcal{S}$, when combined with the non-existence of an inadmissible subset, a sufficient condition for the weak $\mathcal{T}$-admissibility of $\mathcal{S}$, i.e., does the inclusion $\mathfrak{D} \cap \mathfrak{M}_1 \subset \mathfrak{A}$ hold true?

In view of part (ii) in Theorem 7.2.2, we see that if the answer to Question 7.2.2 is positive, then one obtains in fact the equality $\mathfrak{A} = \mathfrak{D} \cap \mathfrak{M}_1$. Hence, in view of part (iii) in Theorem 7.2.2, the answer to Question 7.2.1 would be positive as well.

7.2.5 $\mathcal{T}$-Admissibility of Forward Swaps

We continue working under the assumption that $l = n$. Recall that, by Definition 7.2.6, a family of forward swaps is $\mathcal{T}$-admissible, if it is weakly $\mathcal{T}$-admissible and the solution to the linear system associated with that family is strictly positive for all generic values of $(\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n_+$. In this section, we will give conditions for which a weakly $\mathcal{T}$-admissible family is $\mathcal{T}$-admissible.

The proof of Proposition 7.2.7 follows rather closely the proof of the sufficiency clause in Theorem 1 in Pietersz and Regenmortel [95]. It is important to point out, however, that their result furnishes the sufficient and necessary conditions for the existence of deflated bonds when one is given a family of swaps that enjoys the property that no two swaps start on the same date. The set-up examined in [95] is thus different from ours and the positivity of the deflated bond prices their paper was simply obtained as a by-product of the existence of deflated bond prices up to the bonds maturity dates. One can observe that in fact the assumption of positivity of the deflated bond prices was not used at all in the proof of the necessity clause in Theorem 1 in [95]. For this reason, we need to re-examine the necessity clause. For the reader's convenience, we first give the proof of the sufficiency clause.

Proposition 7.2.7. *Assume that a family $\mathcal{S}$ of forward swaps is weakly $\mathcal{T}$-admissible. If no two swaps start on the same date then a family $\mathcal{S}$ is $\mathcal{T}$-admissible.*

Proof. It is sufficient to demonstrate that a positive solution exists for $b = n$. Assuming that no two forward swaps start on the same date, one can arrange the rows of the matrix C^n so that it becomes an upper triangular matrix with non-zero entries on the diagonal. It is thus clear that the family $\mathcal{S}$ is weakly $\mathcal{T}$-admissible. We will establish the positivity of solution by induction and back substitution. By picking $b = n$, we have $B^n(t, T_n) = 1$, which is strictly positive and greater or equal to 1. Let us assume that $B^n(t, T_j) \geq 1$ for all $j > i$. We would like to show $B^n(t, T_i) \geq 1$ is positive for all generic values of $(\kappa_1, \dots \kappa_n) \in \mathbb{R}^n_+$. By the back substitution algorithm, we obtain

$$B^n(t, T_i) = \kappa_i \sum_{j=i+1}^{m_i} a_j B^n(t, T_j) + B^n(t, T_{m_i}), \quad a_j \geq 0, \quad \forall j = \{i, \dots, m_j\}. \tag{7.14}$$

Next, by the induction hypothesis and the assumption that $(\kappa_1, \dots \kappa_n) \in \mathbb{R}^n_+$, the following holds

$$\kappa_i \sum_{j=i+1}^{m_i} a_j B^n(t, T_j) \geq 0, \qquad B^n(t, T_{m_i}) \geq 1.$$

It now follows easily from (7.14) that $B^n(t, T_i) \geq 1$ for almost all generic values of $(\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n_+$. Note that it was important to assume that the matrix was upper-triangular with non-zero diagonal, since otherwise the back substitution algorithm would not apply. □

Our next goal is to establish a partial converse of Proposition 7.2.7. For this purpose, we need to introduce an additional property of a family $\mathcal{S}$.

Property (A). A family of forward swaps $\mathcal{S}$ is said to satisfy *Property (A)* if for any two forward swaps S_i and S_k from $\mathcal{S}$ such that $T_{s_i} = T_{s_k}$ and $T_{m_k} < T_{m_i}$, we have that $\mathcal{T}(S_k) \subset \mathcal{T}(S_i)$.

The proof of the next result is relegated to the appendix.

Proposition 7.2.8. *If a family $\mathcal{S}$ is $\mathcal{T}$-admissible and satisfies Property (A) then no two swaps in $\mathcal{S}$ starting on the same date exist.*

Let us point out that Property (A) is crucial for the proof of the above proposition. Of course, this assumption is satisfied in the case of standard forward swaps. We now give an example of a family with two forward swaps starting at the same date, for which the solution is strictly positive.

Example 7.2.5. Consider the family $\mathcal{S} = \{S_1 = \{T_0, T_2\}, S_2 = \{T_1, T_3\}, S_3 = \{T_0, T_3\}\}$. Then the associated linear system for $b = 0$ is given by

$$C^0 \overline{x}^0 = \begin{bmatrix} 0 & 1 + \widetilde{a}_2 \kappa_1 & 0 \\ -1 & 0 & 1 + \widetilde{a}_3 \kappa_2 \\ 0 & 0 & 1 + \widetilde{a}_3 \kappa_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \overline{e}^0.$$

It is easy to see that this family of forward swaps does not satisfy Property (A). By solving the above system, we obtain the unique solution

$$\overline{x}^0 = \begin{bmatrix} \frac{1+\widetilde{a}_3\kappa_2}{1+\widetilde{a}_3\kappa_3} \\ (1+\widetilde{a}_2\kappa_1)^{-1} \\ (1+\widetilde{a}_3\kappa_3)^{-1} \end{bmatrix}.$$

From the form of the solution, it is also readily seen that the deflated bonds prices are strictly positive for all generic values of $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3_+$.

7.3 Admissible Market Models of Forward Swap Rates

In this section, we will examine the second issue mentioned in the introduction, that is, the problem whether the joint dynamics of a given family of forward swap rates is uniquely determined. To this end, we will focus on the Radon-Nikodým densities of the corresponding family of *forward swap measures*. The crucial object in the study of forward swap measures is in turn the *swap numéraire*, that is, the denominator appearing in the definition of the forward swap rate. For the detailed analysis of some special cases of models considered in this section, such as: the LIBOR market model and the market model for co-terminal swaps, we refer to Brace et al. [19], Davis and Mataix-Pastor [28], Jamshidian [53, 54, 56], Musiela and Rutkowski [89], Rutkowski [98] (see also, Chapters 12 and 13 in the monograph [90] and the references therein). Due to space limitations, we will not deal here neither with any particular examples of market models of forward swap rates, nor with other related questions, such as the issue of positivity of rates whose dynamics are not directly postulated, but are implicitly given by a specification of a market model for a certain family $\mathcal{S}$ of forward swaps (that is, the out-of-sample swap rates).

7.3.1 Inverse Problem for Swap Annuities

Recall that by a forward swap, we mean the start date T_{s_j}, the maturity date T_{m_j} as well a pair $P^{s_j,m_j}_t, A^{s_j,m_j}$ of processes that are given by (7.2) and (7.3), respectively. The strictly positive process A^{s_j,m_j} is called the *swap numéraire* for the jth forward swap. The forward swap rate κ^j is defined as follows

$$\kappa^j_t = \kappa^{s_j,m_j}_t = \frac{P^{s_j,m_j}_t}{A^{s_j,m_j}_t}, \quad \forall\, t \in [0, T_{s_j}]. \tag{7.15}$$

As before, we consider a finite family $\mathcal{S} = \{S_1, \ldots, S_l\}$ of forward swaps.

Definition 7.3.1. For any $j, d \in \{1, \ldots, l\}$, the *annuity deflated swap numéraire* is given by

$$\widetilde{A}^{d,j}_t := \frac{A^{s_j,m_j}_t}{A^{s_d,m_d}_t}, \quad \forall\, t \in [0, T_{s_j} \wedge T_{s_d}]. \tag{7.16}$$

One should take note here that the *annuity deflated swap numéraire* process $\widetilde{A}^{d,j}$ is in fact the process of our primary interest, since it will later on act as the Radon-Nikodým density process in the specification of the joint dynamics of forward swap rates $\kappa^1, \ldots, \kappa^l$ under a single probability measure. This observation motivates us to formulate the following inverse problem for annuity deflated swap numéraires: provide conditions under which the family of processes $\widetilde{A}^{d,j}$, as specified by equations (7.2), (7.3), (7.15) and (7.16), admit a unique representation in terms of forward swap rates $\kappa^1, \ldots, \kappa^l$ only. In essence, we would like to eliminate bond prices from (7.2), (7.3), (7.15) and (7.16) to express $\widetilde{A}^{d,j}$ in terms of fixed, but arbitrary, parameters $\kappa^1, \ldots, \kappa^l$.

Recall that the *deflated swap annuity* is given by (see (7.7))

$$A^{b,s_j,m_j}_t := \frac{A^{s_j,m_j}_t}{B(t,T_b)} = \sum_{i \in \mathcal{A}_j} \widetilde{a}_i B^b(t, T_i) \tag{7.17}$$

where $\mathcal{A}_j$ is the set of all indices corresponding to settlement dates in the forward swap S_j. Hence (7.16) can also be represented as follows

$$\widetilde{A}_t^{d,j} = \frac{\sum_{i\in\mathcal{A}_j} \widetilde{a}_i B^b(t,T_i)}{\sum_{k\in\mathcal{A}_d} \widetilde{a}_k B^b(t,T_k)} = \frac{A_t^{b,s_j,m_j}}{A_t^{b,s_d,m_d}}, \quad \forall\, t \in [0, T_b \wedge T_{s_j} \wedge T_{s_d}], \tag{7.18}$$

where the second equality is clear since the *deflated swap annuity* A_t^{b,s_j,m_j} is defined by the equality $A_t^{b,s_j,m_j} := A_t^{s_j,m_j}/B(t,T_b)$, for any choice of $b \in \{0,\dots,n\}$. In addition, we write

$$\widetilde{P}_t^{d,j} := \frac{P_t^{s_j,m_j}}{A_t^{s_d,m_d}} = \frac{B(t,T_{s_j}) - B(t,T_{m_j})}{\sum_{k\in\mathcal{A}_d} \widetilde{a}_k B(t,T_k)} = \frac{B^b(t,T_{s_j}) - B^b(t,T_{m_j})}{\sum_{k\in\mathcal{A}_d} \widetilde{a}_k B^b(t,T_k)},$$

so that (7.15) yields, for every $t \in [0, T_b \wedge T_{s_j} \wedge T_{s_d}]$,

$$\kappa_t^j = \kappa_t^{s_j,m_j} = \frac{\widetilde{P}_t^{d,j}}{\widetilde{A}_t^{d,j}} = \frac{B^b(t,T_{s_j}) - B^b(t,T_{m_j})}{\widetilde{A}_t^{d,j} \sum_{k\in\mathcal{A}_d} \widetilde{a}_k B^b(t,T_k)}.$$

Using the notation for generic values of stochastic processes appearing in the formula above, we get

$$\kappa_j = \frac{x_{s_j} - x_{m_j}}{\alpha_{d,j} \sum_{k\in\mathcal{A}_d} \widetilde{a}_k x_k}, \quad j = 1,\dots,l, \tag{7.19}$$

and

$$\alpha_{d,j} = \frac{\sum_{i\in\mathcal{A}_j} \widetilde{a}_i x_i}{\sum_{k\in\mathcal{A}_d} \widetilde{a}_k x_k}, \quad d, j = 1,\dots,l. \tag{7.20}$$

Inverse Problem (IP.2): We say that a family $\mathcal{S} = \{S_1,\dots,S_l\}$ admits a solution to the *inverse problem* (IP.2) if, every $d, j \in \{1,\dots,l\}$, the annuity deflated swap numéraire $\widetilde{A}_t^{d,j}$ can be uniquely expressed as a function of forward swap rates $\kappa_t^1,\dots,\kappa_t^l$. More formally, for any fixed $d \in \{1,\dots,l\}$ and for almost all $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l$, the system of equations (7.19)–(7.20) can be solved for $(\alpha_{d,1},\dots,\alpha_{d,l})$ and a unique solution is given in terms of $(\kappa_1,\dots,\kappa_l)$ only, i.e., it does not depend explicitly on variables $x_0,\dots,x_n$.

The following definition describes the class of families of forward swaps possessing desirable properties from the viewpoint of dynamical properties of a market model.

Definition 7.3.2. We say that a family $\mathcal{S}$ of forward swaps associated with $\mathcal{T}$ is *weakly $\mathcal{A}$-admissible* if for any choice of $d \in \{1,\dots,l\}$ the following property holds: for almost every $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l$ there exists a unique non-zero solution $(\alpha_{d,1},\dots,\alpha_{d,l}) \in \mathbb{R}^l$ to the system of equations (7.19) associated with $\mathcal{S}$ and it is given in terms of $(\kappa_1,\dots,\kappa_l)$ only. If, in addition, the solution is strictly positive for almost every $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l_+$ then we say that a family $\mathcal{S}$ is *$\mathcal{A}$-admissible.*

We will now examine the relationship between the concepts of the (weak) $\mathcal{T}$-admissibility and (weak) $\mathcal{A}$-admissibility. Lemma 7.3.1 shows that the (weak) $\mathcal{T}$-admissibility of a family $\mathcal{S}$ implies its (weak) $\mathcal{A}$-admissibility; the converse does not hold, however, as will be demonstrated by means of a counter-example (see Example 7.3.1).

Lemma 7.3.1. *The (weak) $\mathcal{T}$-admissibility of a family $\mathcal{S}$ implies its (weak) $\mathcal{A}$-admissibility.*

Proof. If the family $\mathcal{S}$ is weakly $\mathcal{T}$-admissible then the uniqueness of a solution to the inverse problem (IP.1) holds and the unique vector of non-zero deflated bond prices $\bar{x}^b$ admits a representation in terms of $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l$. Consequently, in view of (7.18), we conclude that $\mathcal{S}$ is weakly $\mathcal{A}$-admissible. Furthermore, if $\mathcal{S}$ is $\mathcal{T}$-admissible then for almost every $(\kappa_1,\dots,\kappa_l) \in \mathbb{R}^l_+$ the solution to the inverse problem (IP.1) is strictly positive and thus $\alpha_{d,j} = \alpha_{d,j}(\kappa_1,\dots,\kappa_l)$ is strictly positive as well. $\square$

Example 7.3.1. It is worth stressing that the weak $\mathcal{T}$-admissibility is not a necessary condition for the weak $\mathcal{A}$-admissibility of a family $\mathcal{S}$, since it is possible to produce an example of a family of forward swaps, which is weakly $\mathcal{A}$-admissible, but fails to be weakly $\mathcal{T}$-admissible. For this purpose, we set $n = 3$ and we consider the standard forward swaps $S_1 = \{T_0, T_4\}$, $S_2 = \{T_0, T_3\}$ and $S_3 = \{T_2, T_3\}$. The linear system associated with the family $\mathcal{S} = \{S_1, S_2, S_3\}$ for $b = 0$ reads

$$\begin{bmatrix} a_1\kappa_1 & a_2\kappa_1 & a_3\kappa_1 & 1 + a_4\kappa_1 \\ a_1\kappa_2 & a_2\kappa_2 & 1 + a_3\kappa_2 & 0 \\ 0 & 0 & -1 & 1 + a_4\kappa_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

A general solution to the corresponding inverse problem (IP.1) is thus given by

$$\overline{x}^0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{(-a_2\kappa_1\gamma - a_2\kappa_1 a_4\kappa_3\gamma + a_2\kappa_2\gamma + a_2\kappa_1\kappa_2 a_4\gamma) + a_4\kappa_3 - a_3\kappa_1 + a_3\kappa_2 - a_3\kappa_1 a_4\kappa_3 + a_3 a_4\kappa_3\kappa_2 - a_4\kappa_1}{(-\kappa_1 - \kappa_1 a_4\kappa_3 + \kappa_2 + \kappa_2 a_4\kappa_1) a_1} \\ \gamma \\ -\frac{(1 + a_4\kappa_3)(\kappa_1 - \kappa_2)}{-\kappa_1 - \kappa_1 a_4\kappa_3 + \kappa_2 + \kappa_2 a_4\kappa_1} \\ -\frac{\kappa_1 - \kappa_2}{-\kappa_1 - \kappa_1 a_4\kappa_3 + \kappa_2 + \kappa_2 a_4\kappa_1} \end{bmatrix}$$

where γ is a generic value of the deflated bond price $B^0(t, T_2)$. It is thus clear that the family $\mathcal{S}$ is not weakly $\mathcal{T}$-admissible, since the required uniqueness of a solution to the system $C^0\overline{x}^0 = \overline{e}^0$ does not hold.

We will now show that the family $\mathcal{S}$ is weakly $\mathcal{A}$-admissible. To this end, we substitute a solution into the deflated swap annuity equation (7.7) (or, equivalently, (7.17)), to obtain the following expressions for generic values α_{0,s_j,m_j}, $j = 1, 2, 3$ of A_t^{0,s_j,m_j}, $j = 1, 2, 3$

$$\alpha_{0,s_1,m_1} = \frac{a_4\,(-\kappa_3 + \kappa_2)}{-\kappa_1 - \kappa_1 a_4\kappa_3 + \kappa_2 + \kappa_2 a_4\kappa_1},$$

$$\alpha_{0,s_2,m_2} = \frac{a_4\,(-\kappa_3 + \kappa_1)}{-\kappa_1 - \kappa_1 a_4\kappa_3 + \kappa_2 + \kappa_2 a_4\kappa_1},$$

$$\alpha_{0,s_3,m_3} = -\frac{\kappa_1 - \kappa_2}{-\kappa_1 - \kappa_1 a_4\kappa_3 + \kappa_2 + \kappa_2 a_4\kappa_1}.$$

We conclude that the deflated swap annuities can be uniquely expressed as functions of forward swaps rates $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3$ and thus, in view of (7.18), it is now easy to see that the considered family of forward swaps is indeed weakly $\mathcal{A}$-admissible.

Let us now consider a family of forward swaps $\mathcal{S} = \{S_1, \ldots, S_l\}$, which admits a non-zero (but possibly non-unique) solution to the inverse problem (IP.1) for some b. This means that the following equalities hold for some functions $g^{b,k}$ and $g_j^{b,k}$

$$x_k = g^{b,k}(\kappa_1, \ldots, \kappa_l) + \sum_{j=1}^{i} g_j^{b,k}(\kappa_1, \ldots, \kappa_l) x_{n_j}, \quad \forall T_k \in \mathcal{T}, \tag{7.21}$$

where the variables $x_{n_1}, \ldots, x_{n_i}$ correspond to the sub-family of deflated bond prices which parameterize the solution to the linear system associated with $\mathcal{S}$. By substituting equation (7.21) into equation (7.17), we arrive at the following equalities

$$\alpha^{b,s_j,m_j} = \widetilde{g}^{b,j}(\kappa_1, \ldots, \kappa_l) + \sum_{k=1}^{i} \widetilde{g}_k^{b,j}(\kappa_1, \ldots, \kappa_l) x_{n_k} \tag{7.22}$$

for certain functions $\widetilde{g}^{b,j}$ and $\widetilde{g}_k^{b,j}$. It is obvious that, independently of the choice of b, the annuity deflated swap numeráire can now be expressed as follows

$$\alpha_{d,j} = \frac{\widetilde{g}^{b,j}(\kappa_1, \ldots, \kappa_l) + \sum_{k=1}^{i} \widetilde{g}_k^{b,j}(\kappa_1, \ldots, \kappa_l) x_{n_k}}{\widetilde{g}^{b,d}(\kappa_1, \ldots, \kappa_l) + \sum_{k=1}^{i} \widetilde{g}_k^{b,d}(\kappa_1, \ldots, \kappa_l) x_{n_k}}. \tag{7.23}$$

We observe that the family $\mathcal{S}$ is weakly $\mathcal{A}$-admissible, provided that equation (7.23) can be reduced to obtain

$$\alpha_{d,j} = G^{d,j}(\kappa_1, \dots, \kappa_l) \tag{7.24}$$

for some non-zero rational functions $G^{d,j} : \mathbb{R}^l \to \mathbb{R}$. However, if the parameterizing family of bond prices can not be completely eliminated from (7.23) for some annuity deflated swap numéraire $\alpha_{d,j}$ then the family of annuity deflated swap numéraires $\widetilde{A}^{d,j}$, $d, j = 1, \dots, l$ is not uniquely determined by the forward swap rates, which in turn means the family $\mathcal{S}$ of forward swaps fails to be weakly $\mathcal{A}$-admissible.

7.3.2 Dynamics of Forward Swap Rates

We consider here a family of forward swap rate processes $\{\kappa^1, \dots, \kappa^l\}$. Our goal is to show that, under mild conditions imposed on a family of forward swap rate processes, their joint dynamics are uniquely specified by a family of volatility processes with respect to some spanning martingales. We first recall some results from stochastic calculus, which will be used in what follows.

Girsanov's Transforms

Throughout this section, the multidimensional Itô integral should be interpreted as the vector stochastic integral (see, for instance, Shiryaev and Cherny [105]). Let us first quote a general version of the Girsanov theorem (see, for instance, Brémaud and Yor [20] or Theorem 9.4.4.1 in Jeanblanc et al. [61]).

Proposition 7.3.1. *Let $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ be equivalent probability measures on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodým density process*

$$Z_t = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}, \quad \forall\, t \in [0, T]. \tag{7.25}$$

Suppose that M is a $(\mathbb{P}, \mathbb{F})$-local martingale. Then the process

$$\widetilde{M}_t = M_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, M]_s$$

is a $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingale.

Let $\mathcal{M}_{loc}(\mathbb{P})$ ($\mathcal{M}(\mathbb{P})$, resp.) stand for the class of all $(\mathbb{P}, \mathbb{F})$-local martingales ($(\mathbb{P}, \mathbb{F})$-martingales, resp.). Assume that Z is a strictly positive $(\mathbb{P}, \mathbb{F})$-local martingale such that $Z_0 = 1$. Also let an equivalent probability measure $\widetilde{\mathbb{P}}$ be given by (7.25). Then the linear map $\Psi_Z : \mathcal{M}_{loc}(\mathbb{P}) \to \mathcal{M}_{loc}(\widetilde{\mathbb{P}})$ given by the formula

$$\Psi_Z(M) = M_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, M]_s, \quad \forall\, M \in \mathcal{M}_{loc}(\mathbb{P}), \tag{7.26}$$

is called the *Girsanov transform* associated with the Radon-Nikodým density process Z. By the symmetry of the problem, the process Z^{-1} is the Radon-Nikodým density of $\mathbb{P}$ with respect to $\widetilde{\mathbb{P}}$. The corresponding Girsanov transform $\Psi_{Z^{-1}} : \mathcal{M}_{loc}(\widetilde{\mathbb{P}}) \to \mathcal{M}_{loc}(\mathbb{P})$ associated with Z^{-1} is thus given by the formula

$$\Psi_{Z^{-1}}(\widetilde{M}) = \widetilde{M}_t - \int_{(0,t]} Z_s\, d[Z^{-1}, \widetilde{M}]_s, \quad \forall\, \widetilde{M} \in \mathcal{M}_{loc}(\widetilde{\mathbb{P}}).$$

Proposition 7.3.2. *Let $\widetilde{\mathbb{P}}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodým density process Z. Then for any $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingale $\widetilde{N}$ there exists a $(\mathbb{P}, \mathbb{F})$-local martingale $\widehat{N}$ such that*

$$\widetilde{N}_t = \widehat{N}_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, \widehat{N}]_s. \tag{7.27}$$

The process $\widehat{N}$ is given by the formula

$$\widehat{N}_t = \widetilde{N}_0 + \int_{(0,t]} \frac{1}{Z_{s-}}\, dL_s - \int_{(0,t]} \frac{L_{s-}}{Z_{s-}^2}\, dZ_s \tag{7.28}$$

where we denote $L = \widetilde{N}Z$.

From Proposition 7.3.2, it follows immediately that the process $\widehat{N}$ given by (7.28) belongs to the set $(\Psi_Z)^{-1}(\widetilde{N})$. In fact, we have that $\widehat{N} = (\Psi_Z)^{-1}(\widetilde{N})$, as the following well known result shows.

Lemma 7.3.2. *Let $\widetilde{N}$ be any $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingale and let the process $\widehat{N}$ be given by formula (7.28) with $L = \widetilde{N}Z$. Then the process $\widehat{N}$ is also given by the following expression*

$$\widehat{N}_t = \widetilde{N}_t - \int_{(0,t]} Z_s\, d[Z^{-1}, \widetilde{N}]_s. \tag{7.29}$$

The linear map $\Psi_Z : \mathcal{M}_{loc}(\mathbb{P}) \to \mathcal{M}_{loc}(\widetilde{\mathbb{P}})$ is bijective and the inverse map $(\Psi_Z)^{-1} : \mathcal{M}_{loc}(\widetilde{\mathbb{P}}) \to \mathcal{M}_{loc}(\mathbb{P})$ satisfies $(\Psi_Z)^{-1} = \Psi_{Z^{-1}}$.

Let us finally recall the following standard lemma.

Lemma 7.3.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space and let for any $j = 1, \ldots, l$ the process Z^j be a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale with $\mathbb{E}_{\mathbb{P}}(Z_0) = 1$. Then for any fixed $T > 0$ and for each $j = 1, \ldots, l$, there exists a probability measure $\mathbb{P}^j$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodým density process given by*

$$\frac{d\mathbb{P}^j}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t^j, \quad \forall\, t \in [0, T]. \tag{7.30}$$

Construction of a Generic Market Model

We are in a position to examine a general framework under which it is possible to derive the joint dynamics of a family of forward swap rates. In the financial interpretation of the next condition, the processes $\kappa^1, \ldots, \kappa^l$ are aimed to represent forward swap rates for a family of forward swaps $\mathcal{S}$, whereas the processes $Z^1, \ldots, Z^l$ will play the role of Radon-Nikodým densities of swap martingale measures with respect to the underlying probability measure $\mathbb{P}$ (in a practical implementation, $\mathbb{P}$ is typically chosen to be one of the swap martingale measures).

Assumption 7.3.1. We are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We postulate that the $\mathbb{F}$-adapted processes $\kappa^1, \ldots, \kappa^l$ and $Z^1, \ldots, Z^l$ satisfy the following conditions, for every $j = 1, \ldots, l$,
(i) the process Z^j is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale with $\mathbb{E}_{\mathbb{P}}(Z_0^j) = 1$,
(ii) the process $\kappa^j Z^j$ is a $(\mathbb{P}, \mathbb{F})$-martingale,
(iii) the process Z^j is given as a function of some subset of the family $\kappa^1, \ldots, \kappa^l$; specifically, there exists a subset $\{\kappa^{n_1}, \ldots, \kappa^{n_{l_j}}\}$ of the full collection $\{\kappa^1, \ldots, \kappa^l\}$ of forward swap rates and a function $f_j : \mathbb{R}^{l_j} \to \mathbb{R}$ of class C^2 such that $Z^j = f_j(\kappa^{n_1}, \ldots, \kappa^{n_{l_j}})$.

Parts (i) and (ii) in Assumption 7.3.1, when combined with Lemma 7.3.3(i), yield the existence of a family of probability measures $\mathbb{P}^1, \ldots, \mathbb{P}^l$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ for some fixed $T > 0$, such that the process κ^j is a $(\mathbb{P}^j, \mathbb{F})$-martingale for every $j = 1, \ldots, l$, and this in turn implies that κ^j is a $(\mathbb{P}, \mathbb{F})$-semimartingale. Let us remark that the probability measure $\mathbb{P}^j$ is called the jth *swap martingale measure*.

We deduce from part (iii) that the continuous martingale part of Z^j, denoted by $Z^{j,c}$, admits the following integral representation

$$Z_t^{j,c} = Z_0^j + \sum_{i=1}^{l_j} \int_0^t \frac{\partial f_j}{\partial x_i}(\kappa_s^{n_1}, \ldots, \kappa_s^{n_{l_j}})\, d\kappa_s^{n_i,c} \tag{7.31}$$

where $\kappa^{i,c}$ stands for the continuous martingale part of κ^i. To establish equality (7.31), it suffices to apply the Itô formula and use the properties of the stochastic integral.

We will argue that, under the standing Assumption 7.3.1, the semimartingale decomposition of κ^j can be uniquely specified under $\mathbb{P}$ by the choice of the initial values, the volatility processes and the driving martingale, which is henceforth denoted by M. For the purpose of an explicit construction of the model for processes $\kappa^1, \dots, \kappa^l$, we thus start by selecting an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1, \dots, M^k)$ and we define the process κ^j under $\mathbb{P}^j$ as follows, for every $j = 1, \dots, l$,

$$\kappa_t^j = \kappa_0^j + \int_{(0,t]} \kappa_{s-}^j \sigma_s^j \cdot d\Psi^j(M)_s \tag{7.32}$$

where σ^j is the $\mathbb{R}^k$-valued volatility process and the $(\mathbb{P}^j, \mathbb{F})$-martingale $\Psi^j(M)$ equals (see (7.26))

$$\Psi^j(M)_t = M_t - \int_{(0,t]} \frac{1}{Z_s^j}\, d[Z^{j,c}, M^c]_s - \sum_{0<s\le t} \frac{1}{Z_s^j} \Delta Z_s^i \Delta M_s. \tag{7.33}$$

Proposition 7.3.3. *Under Assumption 7.3.1, if the processes $\kappa^1, \dots, \kappa^l$ satisfy (7.32)–(7.33) then for every $j = 1, \dots, l$ the dynamics of κ^j are*

$$\begin{aligned} d\kappa_t^j = {} & \sum_{v=1}^k \kappa_{t-}^j \sigma_t^{j,v}\, dM_t^v - \frac{1}{f_j(\kappa_t^{n_1}, \dots, \kappa_t^{n_{l_j}})} \sum_{i=1}^{l_j} \frac{\partial f_j}{\partial x_i}(\kappa_t^{n_1}, \dots, \kappa_t^{n_{l_j}}) \sum_{v,m=1}^k \kappa_t^j \kappa_t^{n_i} \sigma_t^{j,v} \sigma_t^{n_i,m}\, d[M^{v,c}, M^{m,c}]_t \\ & - \frac{\kappa_{t-}^j}{Z_t^j} \Delta Z_t^j \sum_{v=1}^k \sigma_t^{j,v} \Delta M_t^v. \end{aligned} \tag{7.34}$$

Proof. In view of formulae (7.32) and (7.33), it is clear that the continuous martingale part of κ^j can be expressed as follows

$$\kappa_t^{j,c} = \kappa_0^j + \int_{(0,t]} \kappa_s^j \sigma_s^j \cdot dM_s^c,$$

and thus we infer from (7.31) that the continuous martingale part $Z^{j,c}$ of Z^j admits the following representation

$$Z_t^{j,c} = Z_0^j + \sum_{i=1}^{l_j} \int_{(0,t]} \frac{\partial f_j}{\partial x_i}(\kappa_s^{n_1}, \dots, \kappa_s^{n_{l_j}}) \kappa_s^{n_i} \sigma_s^{n_i} \cdot dM_s^c. \tag{7.35}$$

This allows us to express the process $\Psi^j(M)$ in terms of M, the processes $\kappa^1, \dots, \kappa^n$, their volatilities and the matrix $[M^c]$. To be more specific, we deduce from (7.33) and (7.35) that (note that $[M^{m,c}, M^c]$ is an $\mathbb{R}^k$-valued process)

$$\begin{aligned} \Psi^j(M)_t = M_t & - \int_{(0,t]} \frac{1}{f_j(\kappa_s^{n_1}, \dots, \kappa_s^{n_{l_j}})} \sum_{i=1}^{l_j} \frac{\partial f_j}{\partial x_i}(\kappa_s^{n_1}, \dots, \kappa_s^{n_{l_j}}) \sum_{m=1}^k \kappa_s^{n_i} \sigma_s^{n_i,m}\, d[M^{m,c}, M^c]_s \\ & - \sum_{0<s\le t} \frac{1}{Z_s^j} \Delta Z_s^j \Delta M_s \end{aligned} \tag{7.36}$$

where $Z_s^j = f_j(\kappa_s^{n_1}, \dots, \kappa_s^{n_{l_j}})$ and

$$\Delta Z_s^j = f_j(\kappa_s^{n_1}, \dots, \kappa_s^{n_{l_j}}) - f_j(\kappa_{s-}^{n_1}, \dots, \kappa_{s-}^{n_{l_j}}).$$

By combining (7.32) with (7.36), we arrive at formula (7.34). We thus conclude that, under Assumption 7.3.1, by choosing the driving $(\mathbb{P}, \mathbb{F})$-martingale M and the volatility processes $\sigma^1, \dots, \sigma^l$ in formula (7.32), we completely specify the joint dynamics of processes $\kappa^1, \dots, \kappa^l$ under $\mathbb{P}$. $\square$

Note that when using Proposition 7.3.3 we do not postulate that a model of forward swap rates is already given; in fact, the goal of this result was to describe a general approach to constructing such a model starting from some multi-dimensional driving martingale $M = (M^1, \dots, M^k)$ and arbitrarily chosen $\mathbb{R}^k$-valued volatility processes $\sigma^1, \dots, \sigma^l$ (or course, we should ensure that the stochastic integrals appearing in (7.34) are well defined). Typically, a probability measure $\mathbb{P}$ is selected to be one of the *swap martingale measures* $\mathbb{P}^j$. Therefore, condition (iii) in Assumption 7.3.1 is a consequence of the $\mathcal{A}$-admissibility of a family $\mathcal{S}$. This observation explains why we focused on the inverse problem (IP.2) and the resulting concept of $\mathcal{A}$-admissibility. Intuitively, an $\mathcal{A}$-admissible family of forward swaps is self-consistent, meaning that the joint dynamics of forward swap rate processes $\kappa^1, \dots, \kappa^l$ are entirely determined by the choice of volatilities and driving martingales.

Example 7.3.2. We consider the set-up introduced in Example 7.2.5. Our goal is to specify the joint dynamics of the family of forward swap rates $(\kappa^1, \kappa^2, \kappa^3)$ under a single probability measure. To this end, we assume that we are given a family of volatility processes $\sigma_t^{i,j}$ for $i, j = 1, 2, 3$ and a three-dimensional Brownian motion (W^1, W^2, W^3) defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with the correlation structure is given by $d\langle W^i, W^j\rangle_t = \rho_t^{i,j}\, dt$. We first recall that the family $\mathcal{S}$ considered in Example 7.2.5 was shown to be $\mathcal{T}$-admissible and thus, by Lemma 7.3.1, it is also $\mathcal{A}$-admissible.

The swap annuities in this example are given by: $A_t^{0,2} = \widetilde{a}_2 B(t, T_2)$, $A_t^{1,3} = \widetilde{a}_3 B(t, T_3)$ and $A_t^{0,3} = \widetilde{a}_3 B(t, T_3)$. Hence the annuity deflated swap numéraires for $d = 3$ are given by $\widetilde{A}_t^{3,2} = \widetilde{A}_t^{3,3} = 1$ and

$$\widetilde{A}_t^{3,1} = \frac{\widetilde{a}_2 B(t, T_2)}{\widetilde{a}_3 B(t, T_3)} = \frac{\widetilde{a}_2 B^0(t, T_2)}{\widetilde{a}_3 B^0(t, T_3)}.$$

To derive the joint dynamics of $(\kappa^1, \kappa^2, \kappa^3)$, we postulate that the probability measure $\mathbb{P}$ is such that the process κ^3 is a $(\mathbb{P}, \mathbb{F})$-martingale, so that $\mathbb{P} = \mathbb{P}^3$, and we define the probability measures $d\mathbb{P}^i = Z^i\, d\mathbb{P}$ for $i = 1, 2$, where the Radon-Nikodým density $Z^2 = 1$ and the Radon-Nikodým density process Z^1 is given by

$$Z_t^1 = \frac{1}{c_1} \widetilde{A}_t^{3,1} = \frac{1}{c_1} \frac{A^{0,2}}{A^{1,3}} = \frac{1}{c_1} \frac{\widetilde{a}_2 B^0(t, T_2)}{\widetilde{a}_3 B^0(t, T_3)}$$

where c_1 is a normalizing constant. Using the computations of Example 7.2.5, we see that

$$Z_t^1 = f_1(\kappa_t^1, \kappa_t^2, \kappa_t^3) = \frac{1}{c_1} \frac{\widetilde{a}_2 (1 + \widetilde{a}_3 \kappa_t^3)}{\widetilde{a}_3 (1 + \widetilde{a}_2 \kappa_t^1)},$$

and thus Z^1 is a strictly positive process provided that the forward swap rates κ^1 and κ^3 are strictly positive (the latter property is not obvious a priori, but it will follow from equations (7.37) and (7.38)). To apply equation (7.34), we first compute the first-order partial derivatives

$$\frac{\partial f_1}{\partial x_1}(\kappa_t^1, \kappa_t^2, \kappa_t^3) = \frac{1}{c_1} \frac{\widetilde{a}_2^2 (1 + \widetilde{a}_3 \kappa_t^3)}{\widetilde{a}_3 (1 + \widetilde{a}_2 \kappa_t^1)^2}, \quad \frac{\partial f_1}{\partial x_3}(\kappa_t^1, \kappa_t^2, \kappa_t^3) = \frac{1}{c_1} \frac{\widetilde{a}_2}{(1 + \widetilde{a}_2 \kappa_t^1)}.$$

Using equation (7.34), we conclude that the dynamics of the process κ^1 under the forward swap measure $\mathbb{P}^3$ are given by

$$\begin{aligned} d\kappa_t^1 &= \sum_{i=1}^3 \kappa_t^1 \sigma_t^{1,i}\, dW_t^i + \frac{\widetilde{a}_3 (1 + \widetilde{a}_2 \kappa_t^1)}{\widetilde{a}_2 (1 + \widetilde{a}_3 \kappa_t^3)} \frac{\widetilde{a}_2^2 (1 + \widetilde{a}_3 \kappa_t^3)}{\widetilde{a}_3 (1 + \widetilde{a}_2 \kappa_t^1)^2} \sum_{i,j=1}^3 (\kappa_t^1)^2 \sigma_t^{1,i} \sigma_t^{1,j}\, d\langle W^i, W^j\rangle_t \\ &\quad - \frac{\widetilde{a}_3 (1 + \widetilde{a}_2 \kappa_t^1)}{\widetilde{a}_2 (1 + \widetilde{a}_3 \kappa_t^3)} \frac{\widetilde{a}_2}{(1 + \widetilde{a}_2 \kappa_t^1)} \sum_{i,j=1}^3 \kappa_t^1 \kappa_t^3 \sigma_t^{1,i} \sigma_t^{3,j}\, d\langle W^i, W^j\rangle_t \end{aligned}$$

that is,

$$d\kappa_t^1 = \sum_{i=1}^3 \kappa_t^1 \sigma_t^{1,i}\, dW_t^i + \frac{\widetilde{a}_2}{1+\widetilde{a}_2\kappa_t^1} \sum_{i,j=1}^3 (\kappa_t^1)^2 \sigma_t^{1,i}\sigma_t^{1,j}\rho_t^{i,j}\, dt$$
$$- \frac{\widetilde{a}_3}{1+\widetilde{a}_3\kappa_t^3} \sum_{i,j=1}^3 \kappa_t^1\kappa_t^3 \sigma_t^{1,i}\sigma_t^{3,j}\rho_t^{i,j}\, dt,$$

while the processes κ^2 and κ^3 are governed under $\mathbb{P}^3$ by

$$d\kappa_t^2 = \sum_{i=1}^3 \kappa_t^2\sigma_t^{2,i}\, dW_t^i, \quad d\kappa_t^3 = \sum_{i=1}^3 \kappa_t^3\sigma_t^{3,i}\, dW_t^i. \tag{7.37}$$

The last formula holds, since the Radon-Nikodým density Z^2 is identically one. This peculiar feature comes from the fact that the swaps S_2 and S_3 in Example 7.3.1 only differ by the start date and both have a unique settlement date T_3. Moreover, one can check that the process Z^1 is indeed a strictly positive $(\mathbb{P},\mathbb{F})$-martingale and under $\mathbb{P}^1$ we have that

$$d\kappa_t^1 = \sum_{i=1}^3 \kappa_t^1\sigma_t^{1,i}\, d\widetilde{W}_t^i \tag{7.38}$$

where $(\widetilde{W}^1, \widetilde{W}^2, \widetilde{W}^3)$ is a Brownian motion under $\mathbb{P}^1$ with the same correlation structure as the underlying Brownian motion (W^1, W^2, W^3). We note that for $i = 1,2,3$ the processes κ^i are strictly positive under $\mathbb{P} = \mathbb{P}^3$, since the process κ^i is a stochastic exponential under $\mathbb{P}^i$ for each $i = 1,2,3$, and the probability measures $\mathbb{P}^i$ are equivalent to each other.

Let us conclude by pointing out that weak $\mathcal{A}$-admissible (see Example 7.3.1 for such family) is not sufficient for specification of the joint dynamics since, on the one hand, we can not guarantee that the Radon-Nikodým density is strictly positive and, on the other hand, if one were able to derive some model of the joint dynamics of forward swap rates, it is likely to be inconsistent in the sense that, without additional assumptions, the swap annuities would fail to be positive.

For more details and explicit examples of applications of Proposition 7.3.3 in the context of market models of CDS spreads, the interested reader is referred to Li and Rutkowski [73].

7.4 Appendix

In the appendix, we provide the proofs of Propositions 7.2.3 and 7.2.8.

7.4.1 Proof of Proposition 7.2.3

(i) $\Rightarrow$ (ii). Let us first assume that C^b is diagonalizable. We need to show that there is no $(\mathcal{T},b)$-inadmissible subset $\widetilde{\mathcal{S}}$. Let us consider the diagonalized version of C^b and let us take any subset $\widetilde{\mathcal{S}}$ of $\mathcal{S}$. It the subset has k elements then the number of variables associated with $\widetilde{\mathcal{S}}$ is at least equal to k, since the number of distinct non-zero terms corresponding to $\widetilde{\mathcal{S}}$ that come from the diagonal of C^b equals k. One can also argue that if a vertical block exists in C^b then, by inspection, the matrix cannot be diagonalized.

(ii) $\Rightarrow$ (i). The proof of the implication (ii) $\Rightarrow$ (i) in Lemma 7.2.3, which is presented here, hinges on the inductive argument. An alternative proof, based on a more explicit algorithm for a fixed number of forward swaps, is also available from the authors upon request.

Let us now assume that (ii) holds so that there is no $(\mathcal{T},b)$-inadmissible subset $\widetilde{\mathcal{S}}$ of $\mathcal{S}$. We will proceed by induction. Suppose that the implication (ii) $\Rightarrow$ (i) is true for any $(l-1)\times(n-1)$

matrix. We will argue by contradiction. Suppose that (ii) holds, but C^b cannot be diagonalized. This means that for every non-zero term $c_{1,1},\dots,c_{1,d}$ in the first row, the $(l-1)\times(n-1)$ sub-matrix obtained from C^b by deleting the first row and the column corresponding to non-zero term cannot be diagonalized (without loss of generality, we assume here that the entries $c_{1,1},\dots,c_{1,d}$ in C^b are non-zero and $c_{1,d+1}=\dots=c_{1,n}=0$ for some $d\in\{1,\dots,n\}$. Indeed, if for some non-zero $c_{1,j}$ the $(l-1)\times(n-1)$ sub-matrix can be diagonalized then, obviously, the matrix can be diagonalized as well.

Therefore, by the inductive assumption, for any $i=1,\dots,d$ there exists a subset $\widetilde{\mathcal{S}}_i$ of "shorter" swaps (that is, swaps without the variable x_d) which is not admissible with respect to the reduced set of variables (dates). For each $i=1,\dots,d$, we select the subset $\widetilde{\mathcal{S}}_i$ with the least number of elements. Let k_i be the number of swaps in $\widetilde{\mathcal{S}}_i$. Then the number of the corresponding variables n_i (dates) is less than k_i, that is $n_i\le k_i-1$.

We will argue that by taking the union $\cup_{i=1}^d\widetilde{\mathcal{S}}_i$ of the corresponding "longer" swaps and by adding the first row (i.e., the first "longer" swap) we will produce a $(\mathcal{T},b)$-inadmissible subset $\widetilde{\mathcal{S}}$ of $\mathcal{S}$. This will mean that a contradiction arises since we have assumed that (ii) holds.

Let us first examine the special case when the sets $\widetilde{\mathcal{S}}_i$ are pairwise disjoint. Then the above conclusion is rather obvious, since in the union $\cup_{i=1}^d\widetilde{\mathcal{S}}_i$ the number of forward swaps equals $\sum_{i=1}^d k_i$, whereas the number of variables is given by the following expression

$$\sum_{i=1}^d n_i\le\sum_{i=1}^d k_i-d.$$

By supplementing the first row (i.e., the first swap) we obtain a family $\widetilde{\mathcal{S}}$ with $\sum_{i=1}^d k_i+1$ swaps in which the number of variables is less or equal to $\sum_{i=1}^d k_i$ (we have to add d variables that come from the first swap).

Let us now consider the general case, where an overlap between $\widetilde{\mathcal{S}}_i$ and $\widetilde{\mathcal{S}}_j$ may occur. We will now argue by induction with respect to $i=1,\dots,d$. Suppose that for some $v\le d-1$ the number of swaps in $\cup_{i=1}^v\widetilde{\mathcal{S}}_i$ is greater or equal to the number of variables corresponding to $\cup_{i=1}^v\widetilde{\mathcal{S}}_i$ plus v. We argue that the number of swaps in $\cup_{i=1}^{v+1}\widetilde{\mathcal{S}}_i$ is greater or equal to the number of variables corresponding to $\cup_{i=1}^{v+1}\widetilde{\mathcal{S}}_i$ plus $v+1$. If some swaps from $\widetilde{\mathcal{S}}_{v+1}$ are already in $\cup_{i=1}^v\widetilde{\mathcal{S}}_i$, they do not add any new variables (except for, perhaps, some of the variables $x_1,\dots,x_d$).

Let us then suppose that there are g swaps that are in $\widetilde{\mathcal{S}}_{v+1}$, but not in $\cup_{i=1}^v\widetilde{\mathcal{S}}_i$. The number of new variables corresponding to these swaps is no more than $g-1$; we use here the property that we have chosen the subset $\widetilde{\mathcal{S}}_{v+1}$ with the least number of elements, so that the number of variables corresponding to $k_{v+1}-g$ swaps that are already in $\cup_{i=1}^v\widetilde{\mathcal{S}}_i$ equals at least $k_{v+1}-g$ and the total number of variables corresponding to $\widetilde{\mathcal{S}}_{v+1}$ is no more than $k_{v+1}-1$. Therefore, the number of swaps has grown by g and the number of variables by $g-1$, as desired (plus, perhaps, some of the variables $x_1,\dots,x_d$).

By induction with respect to $i=1,\dots,d$, we conclude that the number of swaps in $\cup_{i=1}^d\widetilde{\mathcal{S}}_i$ is greater of equal to the number of variables corresponding to $\cup_{i=1}^d\widetilde{\mathcal{S}}_i$ plus d. By supplementing the first row (first swap) we obtain a family $\widetilde{\mathcal{S}}$ in which the total number of swaps is strictly greater than the number of variables (dates). This completes the proof of the lemma. □

7.4.2 Proof of Proposition 7.2.8

We work here under the assumption that $l=n$ and we assume that the family $\mathcal{S}$ is $\mathcal{T}$-admissible and satisfies Property (A). This implies, in particular, that $\mathcal{S}$ is weakly $\mathcal{T}$-admissible and the solution to the linear system $C^b\overline{x}^b=\overline{e}^b$ is strictly positive for almost all generic values of $(\kappa_1,\dots,\kappa_n)\in\mathbb{R}^n_+$. As usual, we denote by $\mathcal{A}_i$ the indices of settlement dates of the forward swap S_i and let $\widetilde{A}_i:=\mathcal{A}_i\setminus\{m_i\}$. Our goal is to show that no two forward swaps start on the same date. Suppose, on the contrary, that there exist two forward swaps, S_k and S_i say, starting on the same date, so that $T_{s_k} = T_{s_i} = T_s$ for some s.

By rearranging the forward swap equations generated by S_k and S_i, we obtain

$$B^n(t,T_s) = B^n(t,T_{m_i}) + \kappa_t^i \sum_{j\in\mathcal{A}_i} a_j B^n(t,T_j) \tag{7.39}$$

and

$$B^n(t,T_{m_k}) = \frac{B^n(t,T_s) - \kappa_t^k \sum_{j\in\widetilde{\mathcal{A}}_k} a_j B^n(t,T_j)}{(1+a_{m_k}\kappa_t^k)}. \tag{7.40}$$

After substituting (7.40) into (7.39), we get

$$x_s = x_{m_i}(1+a_{m_i}\kappa_i) + \kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\setminus\{m_k\}} a_j x_j + \frac{a_{m_k}\kappa_i x_s - a_{m_k}\kappa_k\kappa_i \sum_{j\in\widetilde{\mathcal{A}}_k} a_j x_j}{(1+a_{m_k}\kappa_k)}$$

where we use the shorthand notation $\kappa_j = \kappa_t^j$ and $x_j = B^n(t,T_j)$. By rearranging the above equality, we obtain

$$\begin{aligned}
(1+a_{m_k}\kappa_k - a_{m_k}\kappa_i)x_s &= (1+a_{m_k}\kappa_k)(1+a_{m_i}\kappa_i)x_{m_i} + (1+a_{m_k}\kappa_k)\kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\setminus\{m_k\}} a_j x_j \\
&\quad - a_{m_k}\kappa_i\kappa_k \sum_{j\in\widetilde{\mathcal{A}}_k} a_j x_j \\
&= (1+a_{m_i}\kappa_i)(1+a_{m_k}\kappa_k)x_{m_i} + (1+a_{m_k}\kappa_k)\kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\setminus\mathcal{A}_k} a_j x_j \\
&\quad + \kappa_i((1+a_{m_k}\kappa_k) - a_{m_k}\kappa_k) \sum_{j\in\widetilde{\mathcal{A}}_i\cap\widetilde{\mathcal{A}}_k} a_j x_j - a_{m_k}\kappa_i\kappa_k \sum_{j\in\widetilde{\mathcal{A}}_k\setminus\widetilde{\mathcal{A}}_i} a_j x_j \\
&= (1+a_{m_i}\kappa_i)(1+a_{m_k}\kappa_k)x_{m_i} + (1+a_{m_k}\kappa_k)\kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\setminus\mathcal{A}_k} a_j x_j \\
&\quad + \kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\cap\widetilde{\mathcal{A}}_k} a_j x_j - a_{m_k}\kappa_i\kappa_k \sum_{j\in\widetilde{\mathcal{A}}_k\setminus\widetilde{\mathcal{A}}_i} a_j x_j.
\end{aligned}$$

Under the assumption that $\mathcal{T}(S^l) \subset \mathcal{T}(S^i)$, we have that $\widetilde{\mathcal{A}}_i \cap \widetilde{\mathcal{A}}_k = \widetilde{\mathcal{A}}_k$ and

$$a_{m_k}\kappa_i\kappa_k \sum_{j\in\widetilde{\mathcal{A}}_k\setminus\widetilde{\mathcal{A}}_i} a_j x_j = 0.$$

This in turn yields the following equality

$$\begin{aligned}
(1+a_{m_k}\kappa_k - a_{m_k}\kappa_i)x_s &= (1+a_{m_i}\kappa_i)(1+a_{m_k}\kappa_k)x_{m_i} + (1+a_{m_k}\kappa_k)\kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\setminus\mathcal{A}_k} a_j x_j \\
&\quad + \kappa_i \sum_{j\in\widetilde{\mathcal{A}}_k} a_j x_j.
\end{aligned}$$

We thus conclude that x_s is given by

$$x_s = \frac{(1+a_{m_i}\kappa_i)(1+a_{m_k}\kappa_k)x_{m_i} + (1+a_{m_k}\kappa_k)\kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\setminus\mathcal{A}_k} a_j x_j + \kappa_i \sum_{j\in\widetilde{\mathcal{A}}_i\cap\widetilde{\mathcal{A}}_k} a_j x_j}{1+a_{m_k}\kappa_k - a_{m_k}\kappa_i}.$$

Assume that $(\kappa_1,\dots,\kappa_n) \in \mathbb{R}^n_+$ and $x_i \in \mathbb{R}_+$ for every $i \neq s$. Then the deflated bond price $x_s = B^n(t,T_s)$ is strictly positive if and only if $1 + a_{m_k}\kappa_k - a_{m_k}\kappa_i > 0$. This contradicts the assumption that $\mathcal{S}$ is $\mathcal{T}(\mathcal{S})$-admissible, since it is not true that x_s is strictly positive for almost all generic values of $(\kappa_1,\dots,\kappa_n) \in \mathbb{R}^n_+$. □

Chapter 8

Market Model of Forward CDS Spreads

This chapter is based on the paper with the same title by Li and Rutkowski [73] published in *Progress in Probability, Vol. 65, Stochastic Analysis with Financial Applications*, A. Kohatsu-Higa, N. Privault and S.-J. Sheu, eds., Springer, Basel, 2011, pp. 361–411.

8.1 Introduction

The market model for forward LIBORs was first examined in papers by Brace et al. [19] and Musiela and Rutkowski [89]. Their approach was subsequently extended by Jamshidian in [53, 54] to the market model for co-terminal forward swap rates. Since then, several papers on alternative market models for LIBORs and other families of forward swap rates were published. Since modeling of (non-defaultable) forward swap rates is not presented here, the interested reader is referred, for instance, to Galluccio et al. [42], Pietersz and Regenmortel [95], Rutkowski [98], or the monographs by Brace [18] or Musiela and Rutkowski [90] and the references therein.

To the best of our knowledge, there is relatively scarce financial literature in regard either to the existence or to methods of construction of market models for forward CDS spreads. This apparent gap is a bit surprising, especially when confronted with the market practitioners approach to credit default swaptions, which hinges on a suitable variant of the Black formula. The standard argument which underpins the validity of this formula is the postulate of lognormality of credit default swap (CDS) spreads, as discussed, for instance, in Brigo and Morini [24], Jamshidian [55], Morini and Brigo [88], or Rutkowski and Armstrong [99]. In the commonly used *intensity-based approach* to default risk, this crucial property of forward CDS spreads fails to hold, however (see, e.g., Bielecki et al. [13]), and thus a need for a novel modeling approach arises in a natural way.

Recently, attempts have been made to search for explicit constructions of market model for forward CDS spreads in papers by Brigo [21, 22, 23] and Schlögl [101] (related issues were also studied in Lotz and Schlögl [78] and Schönbucher [102]). The present work is inspired by these papers, where under certain simplifying assumptions, the joint dynamics of a family of CDS spreads were derived explicitly under a common probability measure and a construction of the model was provided. Our main goal will be to derive the joint dynamics for certain families of forward CDS spreads in a general semimartingale setup.

Our aim is to derive the joint dynamics of a family of CDS spreads under a common probability. Firstly, we will derive the joint dynamics of a family of single period CDS spreads under the postulate that the interest rate is deterministic. In the second part, we will derive the joint dynamics of a family of single period CDS spreads under the assumption that the interest rate and default indicator process are independent. Lastly, without any simplifying assumptions, we will derive both the joint dynamics of a family of one- and two-period CDS spreads and a family of one-period and co-terminal CDS spreads. We would also like to mention that, although not presented here, the joint dynamics of a family of one-period and co-initial CDS spreads associated with a predetermined tenor structure can also be derived using the techniques developed in Subsections 8.5.3 and 8.5.4.

For each market model, we will also present both the *bottom-up approach* and the *top-down approach* to the modeling of CDS spreads. In the bottom-up approach, one usually starts with a credit risk model and, relying on the assumption that the Predictable Representation Property (PRP) holds, one shows the existence of a family of 'volatility' processes for a family of forward CDS spreads and derives their joint dynamics under a common probability measure. On the other hand, in the top-down approach, for any given in advance family of 'volatility' processes, we focus on the direct derivation of the joint dynamics for a given family of forward CDS spreads under a common probability measure. It is fair to point out, however, that in this chapter we do not provide a fully developed credit risk model obtained through the top-down approach, since the construction of the default time consistent with the derived dynamics of forward CDS spreads is not studied.

In Subsection 8.5.5, we make an attempt to identify the set of postulates that underpin the top-down approach to a generic model of forward spreads and we derive the joint dynamics of forward spreads in a fairly general set-up. We emphasize the fact that this construction is only feasible for a judiciously chosen family of forward spreads. The work concludes by a brief discussion of the most pertinent open problems that need to be addressed in the context of top-down models.

8.2 Forward Credit Default Swaps

Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the *reference filtration*, which is assumed to satisfy the usual conditions. We work throughout within the framework of the *reduced-form* (i.e., intensity-based) methodology. Let us first take the perspective of the *bottom-up approach*, that is, an approach in which we specify explicitly the default time using some salient probabilistic features, such as the knowledge of its *survival process* or, equivalently, the *hazard process*. We thus assume that we are given the default time τ defined on this space in such a way that the $\mathbb{F}$-*survival process* $G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ is positive. It is well known that this goal can be achieved in several alternative ways, for instance, using the so-called canonical construction of the random time for a given in advance $\mathbb{F}$-adapted intensity process λ.

We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ the *full filtration*, that is, the filtration generated by $\mathbb{F}$ and the default indicator process $H_t = \mathbb{1}_{\{\tau \le t\}}$. Formally, we set $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$, where $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ is the filtration generated by H. It is well known that for any $0 \le t < u \le T$ and any $\mathbb{Q}$-integrable, $\mathcal{F}_u$-measurable random variable X the following equality is valid (see, for instance, Chapter 5 in Bielecki and Rutkowski [12] or Chapter 3 in Bielecki et al. [14])

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>u\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, G_t^{-1}\, \mathbb{E}_{\mathbb{Q}}(G_u X \,|\, \mathcal{F}_t). \tag{8.1}$$

Finally, we assume that an underlying default-free term structure model is given and we denote by $\beta(t,u) = B_t B_u^{-1}$ the default-free discount factor over the time period $[t,u]$ for $0 \le t \le u \le T$, where in turn $B = (B_t,\, t \in [0,T])$ represents the savings account. By assumption, the probability measure $\mathbb{Q}$ will be interpreted as the risk-neutral measure. The same basic assumptions underpinning the bottom-up approach will be maintained in Section 8.5 where alternative variants of market models are presented.

Let $\mathcal{T} = \{T_0 < T_1 < \cdots < T_n\}$ with $T_0 \ge 0$ be a fixed *tenor structure* and let us write $a_i = T_i - T_{i-1}$. We observe that it is always true that, for every $i = 1, \ldots, n$,

$$\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_t\right) \ge \mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}_t\right). \tag{8.2}$$

When dealing with the bottom-up approach, we will make the stronger assumption that the following inequality holds, for every $i = 1, \ldots, n$,

$$\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_t\right) > \mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}_t\right). \tag{8.3}$$

We are in a position to formally introduce the concept of the forward credit default swap. To this end, we will describe the cash flows of the two legs of the stylized forward CDS starting at T_i and maturing at T_l, where $T_0 \leq T_i < T_l \leq T_n$. We denote by $\delta_j \in [0, 1)$ the constant recovery rate, which determines the size of the protection payment at time T_j, if default occurs between the dates T_{j-1} and T_j.

Definition 8.2.1. The *forward credit default swap* issued at time $s \in [0, T_i]$, with the unit notional and the $\mathcal{F}_s$-measurable spread κ, is determined by its discounted payoff, which equals $D^{i,l}_t = P^{i,l}_t - \kappa A^{i,l}_t$ for every $t \in [s, T_i]$, where in turn the discounted payoff of the *protection leg* equals

$$P^{i,l}_t = \sum_{j=i+1}^{l} (1 - \delta_j)\beta(t, T_j)\mathbb{1}_{\{T_{j-1} < \tau \leq T_j\}} \tag{8.4}$$

and the discounted payoff of the *fee leg* (also known as the *premium leg*) per one unit of the spread equals

$$A^{i,l}_t = \sum_{j=i+1}^{l} a_j \beta(t, T_j)\mathbb{1}_{\{\tau > T_j\}}. \tag{8.5}$$

Remark 8.2.1. It should be stressed that in the specification of the two legs we have, in particular, deliberately omitted the so-called *accrual payment*, that is, a portion of the fee that should be paid if default occurs between two tenor dates, say T_{j-1} and T_j. The interested reader is referred to Brigo [21, 22, 23], Brigo and Mercurio [25], or Rutkowski [100] for more details. Specifications (8.4)–(8.5) mean that we decided to adopt here the *postponed running* CDS convention proposed by Brigo [21, 22, 23]. This particular choice of convention is motivated by the fact that it appears to be the most convenient for constructing market models of forward CDS spreads.

The value (or the *fair price*) of the forward CDS at time t is based on the risk-neutral formula under $\mathbb{Q}$ applied to the discounted future payoffs. Note that we only consider here the case $t \in [s, T_i]$, although an extension to the general case where $t \in [s, T_l]$ is readily available as well.

Definition 8.2.2. The *value* of the forward credit default swap for the protection buyer equals, for every $t \in [s, T_i]$,

$$S^{i,l}_t(\kappa) = \mathbb{E}_{\mathbb{Q}}(D^{i,l}_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(P^{i,l}_t \,|\, \mathcal{G}_t) - \kappa\, \mathbb{E}_{\mathbb{Q}}(A^{i,l}_t \,|\, \mathcal{G}_t). \tag{8.6}$$

In the second equality in (8.6) we used the definition of the process $D^{i,l}$ and the postulated property that the spread κ is $\mathcal{F}_s$-measurable, and thus also $\mathcal{G}_t$-measurable for every $t \in [s, T_i]$. Let us observe that $A^{i,l}_t = \mathbb{1}_{\{\tau > T_i\}} A^{i,l}_t$ and $P^{i,l}_t = \mathbb{1}_{\{\tau > T_i\}} P^{i,l}_t$ so that also $D^{i,l}_t = \mathbb{1}_{\{\tau > T_i\}} D^{i,l}_t$. Using formula (8.1), it is thus straightforward to show that the value at time $t \in [s, T_i]$ of the forward CDS satisfies

$$S^{i,l}_t(\kappa) = \mathbb{1}_{\{\tau > t\}} G^{-1}_t\, \mathbb{E}_{\mathbb{Q}}(D^{i,l}_t \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau > t\}} \widetilde{S}^{i,l}_t(\kappa), \tag{8.7}$$

where the pre-default price satisfies $\widetilde{S}^{i,l}_t(\kappa) = \widetilde{P}^{i,l}_t - \kappa \widetilde{A}^{i,l}_t$, where we denote

$$\widetilde{P}^{i,l}_t = G^{-1}_t\, \mathbb{E}_{\mathbb{Q}}(P^{i,l}_t \,|\, \mathcal{F}_t), \quad \widetilde{A}^{i,l}_t = G^{-1}_t\, \mathbb{E}_{\mathbb{Q}}(A^{i,l}_t \,|\, \mathcal{F}_t). \tag{8.8}$$

More explicitly, the pre-default value at time $t \in [0, T_i]$ of the *fee leg* per one unit of spread, that is, of the *defaultable annuity* is given by

$$\widetilde{A}^{i,l}_t = \sum_{j=i+1}^{l} a_j G^{-1}_t\, \mathbb{E}_{\mathbb{Q}}(\beta(t, T_j)\mathbb{1}_{\{\tau > T_j\}} \,|\, \mathcal{F}_t). \tag{8.9}$$

Similarly, the pre-default value at time $t \in [0, T_i]$ of the *protection leg* equals

$$\widetilde{P}_t^{i,l} = \sum_{j=i+1}^{l} (1 - \delta_j) G_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(\beta(t, T_j) \mathbb{1}_{\{T_{j-1} < \tau \leq T_j\}} \, | \, \mathcal{F}_t). \tag{8.10}$$

Since the forward CDS is terminated at default, the concept of the *fair* (or *par*) forward CDS spread is only meaningful prior to default. It is also possible, and in fact more convenient, to introduce the notion of the *pre-default* fair forward CDS spread, which can be formally defined for any date $t \in [0, T_i]$. To this end, we use the following definition, in which the issuance date s of a forward CDS is irrelevant, and thus it defines the $\mathbb{F}$-adapted process $\kappa^{i,l} = (\kappa_t^{i,l}, \, t \in [0, T_i])$.

Definition 8.2.3. For any date $t \in [0, T_i]$, the *pre-default fair forward CDS spread* at time t is the $\mathcal{F}_t$-measurable random variable $\kappa_t^{i,l}$ such that $\widetilde{S}_t^{i,l}(\kappa_t^{i,l}) = 0$.

For brevity, the pre-default fair forward CDS spread will simply be called *forward CDS spread* in what follows. Recall that $\widetilde{S}_t^{i,l}(\kappa_t^{i,l}) = \widetilde{P}_t^{i,l} - \kappa_t^{i,l} \widetilde{A}_t^{i,l}$, where the processes $\widetilde{A}^{i,l}$ and $\widetilde{P}^{i,l}$ are given by (8.9) and (8.10), respectively. The forward spread for the CDS starting at T_i and maturing at T_l is thus given as the ratio of pre-default values of the fee and protection legs, that is,

$$\kappa_t^{i,l} = \frac{\widetilde{P}_t^{i,l}}{\widetilde{A}_t^{i,l}}, \quad \forall \, t \in [0, T_i]. \tag{8.11}$$

Within the bottom-up approach, the main goal is to analyze the dynamics of the forward CDS spread $\kappa^{i,l}$ for a given in advance specification of the default time. In particular, we are interested in the existence of a martingale measure for this process and explicit computation of the 'volatility' process for $\kappa^{i,l}$. The latter task appears to be rather difficult and thus it can only be performed for some relatively simple stochastic models of default intensity (see Bielecki et al. [15] who examined the hedging of a credit default swaption in the CIR default intensity model). Moreover, an explicit control of the volatility process for forward CDS spread is not feasible. Therefore, the valuation of options on a forward CDS through closed-form solutions is a very difficult task within the classic reduced-form approach, in which one usually specifies the model by selecting the default intensity process.

By contrast, in the top-down approach we postulate that the volatilities for a given family of forward CDS spreads are predetermined (and thus they can be chosen, for instance, as deterministic functions) and we aim to derive the joint dynamics of these processes under a common probability measure that, typically, is chosen to be a martingale measure for one of these processes. Since this work focuses on the concept of a *market model*, which is obtained in this way, it is clear that we will be mainly interested in the top-down approach. However, a detailed analysis of the bottom-up approach is also useful, since it provides important guidance about the salient features of forward CDS spreads.

Before we proceed to the main topic of this work, we will present and examine certain abstract semimartingale setups that will prove useful in Section 8.5. It is notable that 'abstract' results established within some particular setup may be later used to obtain dynamics within different 'applied' frameworks. For instance, Setup (A) introduced in Subsection 8.4.1 covers simultaneously the cases of (default-free) forward LIBORs under stochastic interest rates and one-period forward CDS spreads under deterministic interest rates.

8.3 Preliminaries

We need first to recall some definitions and results from stochastic calculus, which will be used in what follows. In this section, the multidimensional Itô integral should be interpreted as the vector stochastic integral (see, for instance, Shiryaev and Cherny [105]). We first quote a version of

Girsanov's theorem (see, e.g., Brémaud and Yor [20] or Theorem 9.4.4.1 in Jeanblanc et al. [61]). Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a given filtered probability space. Note that it is not postulated in Section 8.4 that the σ-field $\mathcal{F}_0$ is trivial. For arbitrary real-valued semimartingales X and Y, we denote by $[X, Y]$ their quadratic covariation; it is known to be a finite variation process.

Proposition 8.3.1. *Let $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ be equivalent probability measures on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodým density process*

$$Z_t = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}, \quad \forall\, t \in [0, T]. \tag{8.12}$$

If M is a $(\mathbb{P}, \mathbb{F})$-local martingale then the process

$$\widetilde{M}_t = M_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, M]_s$$

is a $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingale.

Let $\mathcal{M}_{loc}(\mathbb{P}, \mathbb{F})$ ($\mathcal{M}(\mathbb{P}, \mathbb{F})$, respectively) stand for the class of all $(\mathbb{P}, \mathbb{F})$-local martingales ($(\mathbb{P}, \mathbb{F})$-martingales, respectively). Assume that Z is a positive $(\mathbb{P}, \mathbb{F})$-martingale such that $\mathbb{E}_{\mathbb{P}}(Z_0) = 1$. Also, let an equivalent probability measure $\widetilde{\mathbb{P}}$ be given by (8.12). Then the linear map $\Psi_Z : \mathcal{M}_{loc}(\mathbb{P}, \mathbb{F}) \to \mathcal{M}_{loc}(\widetilde{\mathbb{P}}, \mathbb{F})$, which is defined by the formula

$$\Psi_Z(M)_t = M_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, M]_s, \quad \forall\, M \in \mathcal{M}_{loc}(\mathbb{P}, \mathbb{F}),$$

is called the *Girsanov transform* associated with the Radon-Nikodým density process Z.

By the symmetry of the problem, the process $Z^{-1} := 1/Z$ is the Radon-Nikodým density process of $\mathbb{P}$ with respect to $\widetilde{\mathbb{P}}$. The corresponding Girsanov transform $\Psi_{Z^{-1}} : \mathcal{M}_{loc}(\widetilde{\mathbb{P}}, \mathbb{F}) \to \mathcal{M}_{loc}(\mathbb{P}, \mathbb{F})$ associated with Z^{-1} is thus given by the formula

$$\Psi_{Z^{-1}}(\widetilde{M})_t = \widetilde{M}_t - \int_{(0,t]} Z_s\, d[Z^{-1}, \widetilde{M}]_s, \quad \forall\, \widetilde{M} \in \mathcal{M}_{loc}(\widetilde{\mathbb{P}}, \mathbb{F}).$$

Proposition 8.3.2. *Let $\widetilde{\mathbb{P}}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodým density process Z. Then, for any $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingale $\widetilde{N}$, there exists a $(\mathbb{P}, \mathbb{F})$-local martingale $\widehat{N}$ such that*

$$\widetilde{N}_t = \widehat{N}_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, \widehat{N}]_s. \tag{8.13}$$

Let us set $L = \widetilde{N}Z$. Then the process $\widehat{N}$ is given by the formula

$$\widehat{N}_t = \widetilde{N}_0 + \int_{(0,t]} \frac{1}{Z_{s-}}\, dL_s - \int_{(0,t]} \frac{L_{s-}}{Z_{s-}^2}\, dZ_s. \tag{8.14}$$

From Proposition 8.3.2, it follows that the process $\widehat{N}$ given by (8.14) belongs to the set $(\Psi_Z)^{-1}(\widetilde{N})$. In fact, we have that that $\widehat{N} = (\Psi_Z)^{-1}(\widetilde{N})$, as the following result shows.

Lemma 8.3.1. *Let $\widetilde{N}$ be any $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingale and let the process $\widehat{N}$ be given by formula (8.14) with $L = \widetilde{N}Z$. Then the process $\widehat{N}$ is also given by the following expression*

$$\widehat{N}_t = \widetilde{N}_t - \int_{(0,t]} Z_s\, d[Z^{-1}, \widetilde{N}]_s. \tag{8.15}$$

Corollary 8.3.1. *The linear map $\Psi_Z : \mathcal{M}_{loc}(\mathbb{P}, \mathbb{F}) \to \mathcal{M}_{loc}(\widetilde{\mathbb{P}}, \mathbb{F})$ is bijective and the inverse map $(\Psi_Z)^{-1} : \mathcal{M}_{loc}(\widetilde{\mathbb{P}}, \mathbb{F}) \to \mathcal{M}_{loc}(\mathbb{P}, \mathbb{F})$ satisfies $(\Psi_Z)^{-1} = \Psi_{Z^{-1}}$.*

The following definition is standard (see, for instance, Jacod and Yor [52]).

Definition 8.3.1. We say that an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1, \ldots, M^k)$ has the *predictable representation property* (PRP) with respect to $\mathbb{F}$ under $\mathbb{P}$, if an arbitrary $(\mathbb{P}, \mathbb{F})$-local martingale N can be expressed as follows

$$N_t = N_0 + \int_{(0,t]} \xi_s \cdot dM_s, \quad \forall\, t \in [0, T],$$

for some $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable, M-integrable process $\xi = (\xi^1, \ldots, \xi^k)$. We then say that $M = (M^1, \ldots, M^k)$ is the *spanning* $(\mathbb{P}, \mathbb{F})$*-martingale.*

The following well-known result examines the predictable representation property under an equivalent change of a probability measure. For the sake of completeness, we provide the proof of this result.

Proposition 8.3.3. *Assume that the PRP holds under $\mathbb{P}$ with respect to $\mathbb{F}$ with the spanning $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1, \ldots, M^k)$. Let $\widetilde{\mathbb{P}}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodým density process Z. Then the PRP holds under $\widetilde{\mathbb{P}}$ with respect to $\mathbb{F}$ with the spanning $(\widetilde{\mathbb{P}}, \mathbb{F})$-martingale $\widetilde{M} = (\widetilde{M}^1, \ldots, \widetilde{M}^k)$, where for every $l = 1, \ldots, k$,*

$$\widetilde{M}^l_t = M^l_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, M^l]_s.$$

Hence for any $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingale $\widetilde{N}$ there exists an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable, $\widetilde{M}$-integrable process $\widetilde{\xi}$ such that under $\widetilde{\mathbb{P}}$

$$\widetilde{N}_t = \widetilde{N}_0 + \int_{(0,t]} \widetilde{\xi}_s \cdot d\widetilde{M}_s.$$

Proof. By Proposition 8.3.2, there exists $\widehat{N} \in \mathcal{M}_{loc}(\mathbb{P}, \mathbb{F})$ with $\widehat{N}_0 = 0$ such that

$$\widetilde{N}_t - \widetilde{N}_0 = \Psi_Z(\widehat{N})_t = \widehat{N}_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, \widehat{N}]_s.$$

Since the PRP is assumed to hold under $\mathbb{P}$ with respect to $\mathbb{F}$ with the spanning martingale $M = (M^1, \ldots, M^k)$, there exists some $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable, M-integrable process $\xi = (\xi^1, \ldots, \xi^k)$ such that

$$\begin{aligned} \widetilde{N}_t - \widetilde{N}_0 &= \int_{(0,t]} \xi_s \cdot dM_s - \int_{(0,t]} \frac{\xi_s}{Z_s}\, d[Z, M]_s \\ &= \int_{(0,t]} \xi_s \cdot d\Psi_Z(M)_s. \end{aligned}$$

Therefore, the claim holds by setting $\widetilde{\xi} = \xi$ and $\widetilde{M} = \Psi_Z(M)$. Note that the process $\widetilde{\xi}$ is $\widetilde{M}$-integrable under $\widetilde{\mathbb{P}}$, since it is $\Psi_Z(M)$ integrable under $\mathbb{P}$. □

For the definitions and properties of the *stochastic exponential* (also known as the *Doléans-Dade exponential*) and *stochastic logarithm*, we refer to Section 9.4.3 in Jeanblanc et al. [61].

Definition 8.3.2. The *stochastic exponential* $\mathcal{E}(X)$ of a real-valued semimartingale X is the unique solution Y to the following stochastic differential equation

$$Y_t = 1 + \int_{(0,t]} Y_{s-}\, dX_s. \tag{8.16}$$

Let X^c be the *continuous martingale part* of X, that is, the unique continuous local martingale X^c such that $X_0^c = X_0$ and $X - X^c$ is a purely discontinuous semimartingale. Then the unique solution to equation (8.16) is given by the following expression

$$Y_t = \mathcal{E}(X)_t = e^{X_t - X_0 - \frac{1}{2}[X^c, X^c]_t} \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

The map $X \to \mathcal{E}(X)$ can be inverted if almost all paths of the process $\mathcal{E}(X)$ and its left-continuous version $\mathcal{E}(X)_-$ do not hit zero (see, e.g., Choulli et al. [26]). Recall that a subset of $\Omega \times \mathbb{R}_+$ is called *evanescent* if its projection on Ω has null probability.

Let $\mathcal{S}$ be the space of all real-valued semimartingales. Consider the following subspaces:

$$\mathcal{S}_0 = \big\{ X \in \mathcal{S} \,\big|\, \{\Delta X = -1\} \text{ is evanescent} \big\}$$

and

$$\mathcal{S}_1 = \big\{ Y \in \mathcal{S} \,\big|\, \{Y Y_- = 0\} \text{ is evanescent} \big\}.$$

Definition 8.3.3. The *stochastic logarithm* of $Y \in \mathcal{S}_1$ is defined by the formula

$$\mathcal{L}(Y) = \int_{(0,\,\cdot\,]} \frac{1}{Y_{s-}}\, dY_s. \tag{8.17}$$

It is clear that for any non-zero constant semimartingale $Y = c \neq 0$, the stochastic logarithm vanishes, that is, $\mathcal{L}(Y) = \mathcal{L}(c) = 0$. We also have the following result, which shows, in particular, that the map $\mathcal{E} : \mathcal{S}_0 \cap \{X_0 = 0\} \to \mathcal{S}_1 \cap \{Y_0 = 1\}$ is a bijection and its inverse is given by the map $\mathcal{L} : \mathcal{S}_1 \cap \{Y_0 = 1\} \to \mathcal{S}_0 \cap \{X_0 = 0\}$.

Proposition 8.3.4. *(i) For every $X \in \mathcal{S}_0$, the equality $(\mathcal{L} \circ \mathcal{E})(X) = X - X_0$ holds.*
(ii) For every $Y \in \mathcal{S}_1$, the equality $Y = Y_0(\mathcal{E} \circ \mathcal{L})(Y)$ holds.

The following result summarizes the crucial properties of the stochastic logarithm.

Proposition 8.3.5. *(i) For any two semimartingales Y^1 and Y^2 belonging to $\mathcal{S}_1$, the product Y^1Y^2 belongs to $\mathcal{S}_1$ and*

$$\mathcal{L}(Y^1Y^2) = \mathcal{L}(Y^1) + \mathcal{L}(Y^2) + [\mathcal{L}(Y^1), \mathcal{L}(Y^2)]$$

so that

$$\mathcal{L}(Y^1Y^2)^c = \mathcal{L}(Y^1)^c + \mathcal{L}(Y^2)^c. \tag{8.18}$$

(ii) For every $Y \in \mathcal{S}_1$, the process $1/Y$ belongs to $\mathcal{S}_1$ and

$$\mathcal{L}(1/Y) = -\mathcal{L}(Y) + [Y, 1/Y].$$

Therefore,

$$\mathcal{L}(1/Y)^c = -\mathcal{L}(Y)^c. \tag{8.19}$$

(iii) For every $Y \in \mathcal{S}_1$ the continuous martingale part $\mathcal{L}(Y)^c$ of the stochastic logarithm $\mathcal{L}(Y)$ satisfies

$$\mathcal{L}(Y)^c = \left(\int_{(0,\,\cdot\,]} \frac{1}{Y_{s-}}\, dY_s \right)^c = \int_{(0,\,\cdot\,]} \frac{1}{Y_s}\, dY_s^c. \tag{8.20}$$

Consequently, for any continuous local martingale M, we have that

$$[\mathcal{L}(Y)^c, M]_t = \int_{(0,t]} \frac{1}{Y_s}\, d[Y^c, M]_s. \tag{8.21}$$

8.4 Abstract Semimartingale Setups

The goal of this section is to examine a few alternative abstract semimartingale setups and to establish some generic results that will be subsequently used in Section 8.5 to derive the joint dynamics of CDS spreads under various sets of assumptions. In fact, the results of this section can also be used to deal with modeling of non-defaultable forward swap rates, but we do not examine this issue in this work, since it was treated elsewhere (see, for instance, Galluccio et al. [42], Jamshidian [53] or Rutkowski [98]).

Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a given filtered probability space. We consider an $\mathbb{F}$-adapted stochastic processes with the time parameter $t \in [0,T]$ for some fixed $T > 0$. We postulate that the PRP holds with respect to $\mathbb{F}$ under $\mathbb{P}$ with the spanning $(\mathbb{P},\mathbb{F})$-martingale $M = (M^1, \dots, M^k)$.

It is well known (see Proposition 8.3.3) that the PRP is preserved under an equivalent change of measure. Specifically, let $\widetilde{\mathbb{P}}$ be any probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, with the Radon-Nikodým density process

$$Z_t = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}, \quad \forall\, t \in [0,T].$$

Then an arbitrary $(\widetilde{\mathbb{P}},\mathbb{F})$-martingale $\widetilde{N}$ can be expressed under $\widetilde{\mathbb{P}}$ as

$$\widetilde{N}_t = \widetilde{N}_0 + \int_{(0,t]} \widetilde{\xi}_s \cdot d\Psi(M)_s, \quad \forall\, t \in [0,T],$$

for some process $\widetilde{\xi}$, where the $\mathbb{R}^k$-valued $(\widetilde{\mathbb{P}},\mathbb{F})$-martingale $\Psi(M) := (\Psi(M^1), \dots, \Psi(M^k))$ satisfies, for $l = 1, \dots, k$,

$$\Psi(M^l)_t = M^l_t - \int_{(0,t]} \frac{1}{Z_s}\, d[Z, M^l]_s, \quad \forall\, t \in [0,T].$$

The following auxiliary result will prove to be useful in what follows. Of course, the probability measure $\mathbb{P}^i$ introduced in part (i) in Lemma 8.4.1 is simply defined by setting $\mathbb{P}^i(A) = \mathbb{E}_{\mathbb{P}}(Z^i_T \mathbb{1}_A)$ for every $A \in \mathcal{F}_T$. Recall that $M^c = (M^1, \dots, M^k)^c = (M^{1,c}, \dots, M^{k,c})$.

Lemma 8.4.1. *(i) Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let for any $i = 1, \dots, n$ the process Z^i be a positive $(\mathbb{P},\mathbb{F})$-martingale with $\mathbb{E}_{\mathbb{P}}(Z^i_0) = 1$. Then, for each $i = 1, \dots, n$, there exists a probability measure $\mathbb{P}^i$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodým density process given by*

$$\frac{d\mathbb{P}^i}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z^i_t, \quad \forall\, t \in [0,T]. \tag{8.22}$$

(ii) Let us assume, in addition, that the PRP holds with respect to $\mathbb{F}$ under $\mathbb{P}$ with the spanning $(\mathbb{P},\mathbb{F})$-martingale $M = (M^1, \dots, M^k)$. Then, for any $i = 1, \dots, n$, the spanning $(\mathbb{P}^i,\mathbb{F})$-martingale $\Psi^i(M) = (\Psi^i(M^1), \dots, \Psi^i(M^k))$ is given by the following expression, for every $l = 1, \dots, k$,

$$\Psi^i(M^l)_t = M^l_t - \int_{(0,t]} \frac{1}{Z^i_s}\, d[Z^i, M^l]_s \tag{8.23}$$

or, equivalently,

$$\Psi^i(M)_t = M_t - [\mathcal{L}(Z^i)^c, M^c]_t - \sum_{0<s\le t} \frac{1}{Z^i_s} \Delta Z^i_s \Delta M_s. \tag{8.24}$$

The equality $\Psi^i(M)^c = M^c$ holds for every $i = 1, \dots, n$.

Proof. The first part of the lemma is rather clear. The second part in Lemma 8.4.1 follows from Proposition 8.3.1 with $\widetilde{\mathbb{P}} = \mathbb{P}^i$ except for equality (8.24). To derive (8.24) from (8.23), we start by recalling that

$$[Z^i, M^l]_t = [Z^{i,c}, M^{l,c}]_t + \sum_{0<s\le t} \Delta Z^i_s \Delta M^l_s.$$

By combining this equality with (8.23), we obtain

$$\Psi^i(M^l)_t = M^l_t - \int_{(0,t]} \frac{1}{Z^i_s}\, d[Z^{i,c}, M^{l,c}]_s - \sum_{0<s\le t} \frac{1}{Z^i_s} \Delta Z^i_s \Delta M^l_s.$$

Formula (8.21) yields, for every $l = 1, \dots, k$,

$$[\mathcal{L}(Z^i)^c, M^{l,c}]_t = \int_{(0,t]} \frac{1}{Z^i_s}\, d[Z^{i,c}, M^{l,c}]_s,$$

and thus (8.24) follows. The equality $\Psi^i(M)^c = M^c$ is an immediate consequence of (8.24). □

In what follows, we will write $[M^c]$ and to denote the matrix of quadratic variation processes for the $\mathbb{R}^k$-valued process M^c, that is,

$$[M^c] = \begin{bmatrix} [M^{1,c}, M^{1,c}] & \dots & [M^{1,c}, M^{k,c}] \\ \vdots & \ddots & \vdots \\ [M^{k,c}, M^{1,c}] & \dots & [M^{k,c}, M^{k,c}] \end{bmatrix}. \tag{8.25}$$

8.4.1 Setup (A)

Throughout this subsection, we will work under the following standing assumptions, referred to as Setup (A) in the sequel. Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a given filtered probability space.

Assumption 8.4.1. We postulate that:
(i) the process $X = (X^1, \dots, X^n)$ is $\mathbb{F}$-adapted,
(ii) the processes $1 + a_j X^j$, $j = 1, \dots, n$ belong to $\mathcal{S}_1$, where $a_1, \dots, a_n$ are non-zero constants,
(iii) for every $i = 0, \dots, n$, the process $Z^{X,i}$, which is given by the formula (by the usual convention, $Z^{X,n} = 1$)

$$Z^{X,i}_t = \prod_{j=i+1}^{n} (1 + a_j X^j_t), \tag{8.26}$$

is a positive $(\mathbb{P}, \mathbb{F})$-martingale,(iv) the PRP holds under $(\mathbb{P}, \mathbb{F})$ with the spanning $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1, \dots, M^k)$.

The following lemma reveals an interesting feature of Setup (A).

Lemma 8.4.2. *For every $i = 1, \dots, n$, the process $Z^{X,i} X^i$ is a $(\mathbb{P}, \mathbb{F})$-martingale.*

Proof. We first observe that $Z^{X,n-1}_t = 1 + a_n X^n_t$ where a_n is a non-zero constant. Hence X^n is a $(\mathbb{P}, \mathbb{F})$-martingale and, since $Z^{X,n} = 1$, this also means that $Z^{X,n} X^n$ is a $(\mathbb{P}, \mathbb{F})$-martingale. Note also that (8.26) yields, for $i = 1, \dots, n-2$,

$$Z^{X,i}_t = (1 + a_{i+1} X^{i+1}_t) Z^{X,i+1}_t = Z^{X,i+1}_t + a_{i+1} X^{i+1}_t Z^{X,i+1}_t,$$

where, by assumption, the processes $Z^{X,i}$ and $Z^{X,i+1}$ are $(\mathbb{P}, \mathbb{F})$-martingales. Since a_{i+1} is a non-zero constant, we conclude that the process $Z^{X,i+1}_t X^{i+1}_t$ is a $(\mathbb{P}, \mathbb{F})$-martingale as well. □

Let the probability measures $\mathbb{P}^i$, $i = 1, \dots, n$ be defined as in Lemma 8.4.1, with the Radon-Nikodým density processes Z^i, $i = 1, \dots, n$ given by

$$Z^i_t = c_i Z^{X,i}_t = c_i \prod_{j=i+1}^{n} (1 + a_j X^j_t), \tag{8.27}$$

where the positive constants $c_1, \dots, c_n$ are chosen in such a way that $\mathbb{E}_{\mathbb{P}}(Z^i_0) = 1$ for $i = 1, \dots, n$. Note that, in particular, we have that $\mathbb{P}^n = \mathbb{P}$ since $Z^n = 1$.

Lemma 8.4.3. *For every $i = 1, \dots, n$, the process X^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale and it admits the following representation under $\mathbb{P}^i$*

$$X_t^i = X_0^i + \int_{(0,t]} \xi_s^i \cdot d\Psi^i(M)_s = X_0^i + \int_{(0,t]} \sum_{l=1}^k \xi_s^{i,l}\, d\Psi^i(M^l)_s, \tag{8.28}$$

where $\xi^i = (\xi^{i,1}, \dots, \xi^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process and the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ is given by (8.24), so that the equality $\Psi^i(M)^c = M^c$ holds for every $i = 1, \dots, n$.

Proof. Let us fix $i = 1, \dots, n$. It follows easily from Lemma 8.4.2 that the process X^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale. Furthermore, in view of part (ii) in Lemma 8.4.1, the PRP holds with respect to $\mathbb{F}$ under $\mathbb{P}^i$ with the spanning $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$. This immediately implies the stated property. □

Remark 8.4.1. It is worth stressing that part (iv) in Assumptions 8.4.2 is only used to deduce the existence of processes $\xi^1, \dots, \xi^n$ such that representation (8.28) holds for processes $X^1, \dots, X^n$. In Section 8.5, when dealing with the construction of market models, we will not assume that the PRP holds, and we will postulate instead that for a given family of forward CDS spreads the counterpart of representation (8.28) is valid for some 'volatility' processes, which can be chosen arbitrarily. A similar remark applies to other abstract setups considered in the foregoing subsections.

The following result underpins, on the one hand, the market model of forward LIBORs (see, for instance, Jamshidian [54]) and, on the other hand, the market model of one-period forward CDS spreads under the assumption that interest rates are deterministic (see Section 8.5.1 below). In the proof of Proposition 8.4.1, part (iv) in Assumptions 8.4.2 is only used indirectly through formula (8.28).

Proposition 8.4.1. *For every $i = 1, \dots, n$, the semimartingale decomposition of the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ under the probability measure $\mathbb{P}^n = \mathbb{P}$ is given by*

$$\Psi^i(M)_t = M_t - \sum_{j=i+1}^n \int_{(0,t]} \frac{a_j \xi_s^j \cdot d\,[M^c]_s}{1 + a_j X_s^j} - \sum_{0<s\le t} \frac{1}{Z_s^i}\, \Delta Z_s^i \Delta M_s. \tag{8.29}$$

More explicitly, for every $i = 1, \dots, n$ and $l = 1, \dots, k$,

$$\begin{aligned}\Psi^i(M^l)_t = M_t^l &- \sum_{j=i+1}^n \int_{(0,t]} \frac{a_j}{1 + a_j X_s^j} \sum_{m=1}^k \xi_s^{j,m}\, d[M^{l,c}, M^{m,c}]_s \\ &- \sum_{0<s\le t} \frac{1}{Z_s^i}\, \Delta Z_s^i \Delta M_s^l.\end{aligned} \tag{8.30}$$

For every $i = 1, \dots, n$, the dynamics of the process X^i with respect to the $(\mathbb{P}, \mathbb{F})$-martingale M are

$$\begin{aligned}dX_t^i = \sum_{l=1}^k \xi_t^{i,l}\, dM_t^l &- \sum_{j=i+1}^n \frac{a_j}{1 + a_j X_t^j} \sum_{l,m=1}^k \xi_t^{i,l} \xi_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\ &- \frac{1}{Z_t^i}\, \Delta Z_t^i \sum_{l=1}^k \xi_t^{i,l} \Delta M_t^l.\end{aligned} \tag{8.31}$$

Proof. We will apply part (ii) of Lemma 8.4.1. We start by noting that (8.17), (8.18), (8.20) and (8.26) yield

$$\mathcal{L}(Z^i)_t^c = \sum_{j=i+1}^n \mathcal{L}(1 + a_j X^j)_t^c = \sum_{j=i+1}^n \left(\int_{(0,\,\cdot\,]} \frac{a_j\, dX_s^j}{1 + a_j X_{s-}^j} \right)_t^c = \sum_{j=i+1}^n \int_{(0,t]} \frac{a_j\, dX_s^{j,c}}{1 + a_j X_s^j},$$

where we also used part (ii) in Assumptions 8.4.1. In view of (8.24) and (8.28), we have that

$$\int_{(0,t]} \frac{a_j\, dX^{j,c}_s}{1+a_jX^j_s} = \int_{(0,t]} \frac{a_j\xi^j_s}{1+a_jX^j_s} \cdot d\Psi^j(M)^c_s = \int_{(0,t]} \frac{a_j\xi^j_s}{1+a_jX^j_s} \cdot dM^c_s,$$

where the second equality follows from the fact that the equality $\Psi^j(M)^c = M^c$ holds for every $j = 1,\dots,n$ (see Lemma 8.4.1). Consequently,

$$[\mathcal{L}(Z^i)^c, M^c]_t = \sum_{j=i+1}^{n} [\int_{(0,\,\cdot\,]} \frac{a_j\xi^j_s}{1+a_jX^j_s} \cdot dM^c_s,\, M^c]_t = \sum_{j=i+1}^{n} \int_{(0,t]} \frac{a_j\xi^j_s \cdot d[M^c]_s}{1+a_jX^j_s}.$$

Combining the last formula with formula (8.24) in Lemma 8.4.1, we obtain the desired equality (8.29). Formula (8.30) is an immediate consequence of (8.29). Finally, in order to derive the dynamics of X^i under $\mathbb{P}$, it suffices to combine (8.28) with (8.30). $\square$

8.4.2 Setup (B)

In this subsection, we work under the following extension of Assumptions 8.4.1, which is hereafter termed Setup (B). We assume that we are given two independent filtrations, denoted as $\mathbb{F}^1$ and $\mathbb{F}^2$, and we set $\mathbb{F} = \mathbb{F}^1 \vee \mathbb{F}^2$, meaning that for every t we have that $\mathcal{F}_t = \sigma(\mathcal{F}^1_t, \mathcal{F}^2_t)$. We consider the $\mathbb{R}^n$-valued processes X and Y given on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$.

Assumption 8.4.2. We postulate that:
(i) the processes $X = (X^1,\dots,X^n)$ and $Y = (Y^1,\dots,Y^n)$ are adapted to the filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$, respectively,
(ii) the processes $1 + a_jX^j$, $j = 1,\dots,n$ and $1 + b_jY^j$, $j = 1,\dots,n$ belong to $\mathcal{S}_1$, where $a_1,\dots,a_n$ and $b_1,\dots,b_n$ are non-zero constants,
(iii) for every $i = 0,\dots,n$, the processes $Z^{X,i}$ and $Z^{Y,i}$, which are given by (by convention, $Z^{X,n} = Z^{Y,n} = 1$)

$$Z^{X,i}_t = \prod_{j=i+1}^{n} (1 + a_jX^j_t),$$
$$Z^{Y,i}_t = \prod_{j=i+1}^{n} (1 + b_jY^j_t),$$

are positive $(\mathbb{P}, \mathbb{F})$-martingales,
(iv) the PRP holds under $(\mathbb{P}, \mathbb{F})$ with the spanning $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1,\dots,M^k)$.

The following auxiliary lemma is rather standard and thus its proof is omitted.

Lemma 8.4.4. *Assume that $\mathbb{F}^1$ and $\mathbb{F}^2$ are independent filtrations and let $\mathbb{F} = \mathbb{F}^1 \vee \mathbb{F}^2$.*
(i) Let X and Y be real-valued stochastic processes adapted to the filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$, respectively. Then we have that, for every $0 \le s \le t$,

$$\mathbb{E}_{\mathbb{P}}(X_tY_t \,|\, \mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(X_t \,|\, \mathcal{F}_s)\, \mathbb{E}_{\mathbb{P}}(Y_t \,|\, \mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(X_t \,|\, \mathcal{F}^1_s)\, \mathbb{E}_{\mathbb{P}}(Y_t \,|\, \mathcal{F}^2_s).$$

(ii) Let M and N be real-valued martingales with respect to the filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$, respectively. Then the product MN is a $(\mathbb{P}, \mathbb{F})$-martingale.

Remark 8.4.2. It is enough to assume in condition (iii) in Assumptions 8.4.2 that the process $Z^{X,i}$ ($Z^{Y,i}$, respectively) is a $(\mathbb{P}, \mathbb{F}^1)$-martingale (a $(\mathbb{P}, \mathbb{F}^2)$-martingale, respectively). Indeed, if the process $Z^{X,i}$ is a $(\mathbb{P}, \mathbb{F}^1)$-martingale then it also follows a $(\mathbb{P}, \mathbb{F})$-martingale, by the assumed independence of filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$. Of course, the symmetric argument can be applied to the process $Z^{Y,i}$.

We are in a position to prove the following generalization of Lemma 8.4.2.

Lemma 8.4.5. *Under Assumptions 8.4.2 the following properties hold, for every $i = 1, \ldots, n$,*
(i) the processes $Z^{Y,i}Z^{X,i}X^i$ and $Z^{Y,i}Z^{X,i}Y^i$ are $(\mathbb{P}, \mathbb{F})$-martingales,
(ii) let the process $Z^{X,Y,i}$ be defined by the formula

$$Z_t^{X,Y,i} := c_i Z_t^{X,i} Z_t^{Y,i} = c_i \prod_{j=i+1}^{n} (1 + a_j X_t^j)(1 + b_j Y_t^j), \tag{8.32}$$

where c_i is a positive constant; then $Z^{X,Y,i}$ is a positive $(\mathbb{P}, \mathbb{F})$-martingale.

Proof. To prove part (i), we first argue, as in the proof of Lemma 8.4.2, that the processes $Z^{X,i}X^i$ and $Z^{Y,i}Y^i$ are $(\mathbb{P}, \mathbb{F})$-martingales. Since $Z^{X,i}X^i$ ($Z^{Y,i}Y^i$, respectively) is $\mathbb{F}^1$-adapted ($\mathbb{F}^2$-adapted, respectively) we conclude that $Z^{X,i}X^i$ ($Z^{Y,i}Y^i$, respectively) is also a $(\mathbb{P}, \mathbb{F}^1)$-martingale ($(\mathbb{P}, \mathbb{F}^2)$-martingale, respectively). We also observe that the process $Z^{Y,i}$ is a $(\mathbb{P}, \mathbb{F}^2)$-martingale and thus, in view of part (ii) in Lemma 8.4.4, we conclude that the product $Z^{Y,i}Z^{X,i}X^i$ is a $(\mathbb{P}, \mathbb{F})$-martingale. By the same token, the product $Z^{X,i}Z^{Y,i}Y^i$ is a $(\mathbb{P}, \mathbb{F})$-martingale. The proof of the second statement is based on similar arguments and thus it is omitted. □

In what follows, it is assumed that the constants c_i are chosen in such a way that $\mathbb{E}_{\mathbb{P}}(Z_0^{X,Y,i}) = 1$ for every $i = 1, \ldots, n$. Let the probability measures $\mathbb{P}^i$, $i = 1, \ldots, n$ be defined as in part (i) in Lemma 8.4.1 with the Radon-Nikodým density processes $Z^i = Z^{X,Y,i}$. In particular, we have that $\mathbb{P}^n = \mathbb{P}$ since, by the usual convention, $Z^{X,Y,n} = 1$. The following result is a slight extension of part (ii) in Lemma 8.4.1.

Lemma 8.4.6. *For every $i = 1, \ldots, n$, the processes X^i and Y^i are $(\mathbb{P}^i, \mathbb{F})$-martingales and they have the following integral representations under $\mathbb{P}^i$*

$$X_t^i = X_0^i + \int_{(0,t]} \xi_s^i \cdot d\Psi^i(M)_s, \tag{8.33}$$

and

$$Y_t^i = Y_0^i + \int_{(0,t]} \zeta_s^i \cdot d\Psi^i(M)_s, \tag{8.34}$$

where $\xi^i = (\xi^{i,1}, \ldots, \xi^{i,k})$ and $\zeta^i = (\zeta^{i,1}, \ldots, \zeta^{i,k})$ are $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable processes and the $\mathbb{R}^k$-valued $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ is given by.

$$\Psi^i(M)_t = M_t - [\mathcal{L}(Z^{X,Y,i})^c, M^c]_t - \sum_{0<s\le t} \frac{1}{Z_s^{X,Y,i}} \Delta Z_s^{X,Y,i} \Delta M_s. \tag{8.35}$$

The equality $\Psi^i(M)^c = M^c$ holds for every $i = 1, \ldots, n$.

Proof. In view of Lemma 8.4.5, the processes X^i and Y^i are manifestly $(\mathbb{P}^i, \mathbb{F})$-martingales. Moreover, by virtue of part (ii) in Lemma 8.4.1, the PRP holds with respect to $\mathbb{F}$ under $\mathbb{P}^i$ with the spanning $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$. This in turn implies that equalities (8.33) and (8.34) hold for some suitably integrable processes ξ^i and ζ^i. Equality (8.35) follows immediately from (8.24). The last statement is a consequence of (8.35). □

The following result will be employed in the joint modeling of forward LIBORs and forward CDS spreads.

Proposition 8.4.2. *Suppose that Assumptions 8.4.2 are satisfied. Then for every $i = 1, \ldots, n$ the semimartingale decomposition of the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ under the probability measure $\mathbb{P}^n = \mathbb{P}$ is given by the following expression*

$$\begin{aligned}\Psi^i(M)_t = M_t &- \sum_{j=i+1}^{n} \int_{(0,t]} \frac{a_j \xi_s^j \cdot d[M^c]_s}{1 + a_j X_s^j} - \sum_{j=i+1}^{n} \int_{(0,t]} \frac{b_j \zeta_s^j \cdot d[M^c]_s}{1 + b_j Y_s^j} \\ &- \sum_{0<s\le t} \frac{1}{Z_s^{X,Y,i}} \Delta Z_s^{X,Y,i} \Delta M_s.\end{aligned} \tag{8.36}$$

The dynamics of X^i and Y^i with respect to the $(\mathbb{P},\mathbb{F})$-martingale M are given by

$$\begin{aligned} dX^i_t &= \sum_{l=1}^k \xi^{i,l}_t \, dM^l_t - \sum_{j=i+1}^n \frac{a_j}{1+a_jX^j_t} \sum_{l,m=1}^k \xi^{i,l}_t \xi^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\ &\quad - \sum_{j=i+1}^n \frac{b_j}{1+b_jY^j_t} \sum_{l,m=1}^k \xi^{i,l}_t \zeta^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t - \frac{1}{Z^{X,Y,i}_t} \, \Delta Z^{X,Y,i}_t \sum_{l=1}^k \xi^{i,l}_t \Delta M^l_t \end{aligned}$$

and

$$\begin{aligned} dY^i_t &= \sum_{l=1}^k \zeta^{i,l}_t \, dM^l_t - \sum_{j=i+1}^n \frac{a_j}{1+a_jX^j_t} \sum_{l,m=1}^k \zeta^{i,l}_t \xi^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\ &\quad - \sum_{j=i+1}^n \frac{b_j}{1+b_jY^j_t} \sum_{l,m=1}^k \zeta^{i,l}_t \zeta^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t - \frac{1}{Z^{X,Y,i}_t} \, \Delta Z^{X,Y,i}_t \sum_{l=1}^k \zeta^{i,l}_t \Delta M^l_t. \end{aligned}$$

Proof. The proof is analogous to the proof of Proposition 8.4.1. We start by noting that

$$\mathcal{L}(Z^{X,Y,i})^c_t = \sum_{j=i+1}^n \mathcal{L}(1+a_jX^j)^c_t + \sum_{j=i+1}^n \mathcal{L}(1+b_jY^j)^c_t$$

$$= \sum_{j=i+1}^n \int_{(0,t]} \frac{a_j \, dX^{j,c}_s}{1+a_jX^j_s} + \sum_{j=i+1}^n \int_{(0,t]} \frac{b_j \, dY^{j,c}_s}{1+b_jY^j_s}.$$

Recall that $\Psi^j(M)^c = M^c$ for every $j = 1, \dots, n$. In view of (8.33) and (8.34), by similar computations as in the proof of Proposition 8.4.1, we thus obtain

$$\begin{aligned} [\mathcal{L}(Z^{X,Y,i})^c, \, M^c]_t &= \sum_{j=i+1}^n [\int_{(0,\,\cdot\,]} \frac{a_j \xi^j_s}{1+a_jX^j_s} \cdot dM^c_s, \, M^c]_t \\ &\quad + \sum_{j=i+1}^n [\int_{(0,\,\cdot\,]} \frac{b_j \zeta^j_s}{1+b_jY^j_s} \cdot dM^c_s, \, M^c]_t \end{aligned}$$

and this in turn yields

$$[\mathcal{L}(Z^{X,Y,i})^c, M^c]_t = \sum_{j=i+1}^n \int_{(0,t]} \frac{a_j \xi^j_s \cdot d[M^c]_s}{1+a_jX^j_s} + \sum_{j=i+1}^n \int_{(0,t]} \frac{b_j \zeta^j_s \cdot d[M^c]_s}{1+b_jY^j_s}. \tag{8.37}$$

By combining (8.35) with (8.37), we obtain (8.36). To establish the dynamics of processes X^i and Y^i under $\mathbb{P}$, it suffices to substitute (8.36) into (8.33) and (8.34), respectively. □

8.4.3 Setup (C)

Throughout this subsection, we will work under the following set of assumptions, which are satisfied by the processes X and Y defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$.

Assumption 8.4.3. We postulate that:
(i) the processes $X = (X^1, \dots, X^n)$ and $Y = (Y^2, \dots, Y^n)$ are $\mathbb{F}$-adapted,
(ii) for every $i = 1, \dots, n$, the process $Z^{X,i}$, which is given by the formula (by the usual convention, $Z^{X,n} = 1$)

$$Z^{X,i}_t = c_i \prod_{j=i+1}^n \frac{Y^j_t - X^j_t}{X^{j-1}_t - Y^j_t}, \tag{8.38}$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale and $c_1,\dots,c_n$ are constants such that $\mathbb{E}_{\mathbb{P}}(Z_0^{X,i})=1$,
(iii) for every $i=2,\dots,n$, the process $Z^{Y,i}$, which is given by the formula

$$Z_t^{Y,i}=\widetilde{c}_i\big(Z_t^{X,i}+Z_t^{X,i-1}\big)=\widetilde{c}_i\,\frac{X_t^{i-1}-X_t^i}{X_t^{i-1}-Y_t^i}\,Z_t^{X,i}, \tag{8.39}$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale and $\widetilde{c}_2,\dots,\widetilde{c}_n$ are constants such that $\mathbb{E}_{\mathbb{P}}(Z_0^{Y,i})=1$,
(iv) for every $i=1,\dots,n$, the process X^i is a $(\mathbb{P}^i,\mathbb{F})$-martingale, where the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{X,i}$,
(v) for every $i=2,\dots,n$, the process Y^i is a $(\widetilde{\mathbb{P}}^i,\mathbb{F})$-martingale, where the Radon-Nikodým density of $\widetilde{\mathbb{P}}^i$ with respect to $\mathbb{P}$ equals $Z^{Y,i}$,
(vi) the PRP holds under $(\mathbb{P},\mathbb{F})$ with the spanning $(\mathbb{P},\mathbb{F})$-martingale $M=(M^1,\dots,M^k)$.

When dealing with Setups (A) and (B), we explicitly stated the assumption that certain processes are in the space $\mathcal{S}_1$, so that their stochastic logarithms are well defined. In the case of Setups (C) and (D), we make instead a general assumption that whenever the notion of the stochastic logarithm is employed, the underlying stochastic process is assumed to belong to the space $\mathcal{S}_1$.

Note also that the probability measures $\mathbb{P}^i$ and $\widetilde{\mathbb{P}}^i$ are defined as in part (i) in Lemma 8.4.1, with the Radon-Nikodým density processes $Z^{X,i}$ and $Z^{Y,i}$, respectively. In particular, we have that $\mathbb{P}^n=\mathbb{P}$ since $Z^{X,n}=1$.

Lemma 8.4.7. *For every $i=1,\dots,n$, the process X^i admits the following representation under $\mathbb{P}^i$*

$$X_t^i=X_0^i+\int_{(0,t]}\xi_s^i\cdot d\Psi^i(M)_s, \tag{8.40}$$

where $\xi^i=(\xi^{i,1},\dots,\xi^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process and the $\mathbb{R}^k$-valued $(\mathbb{P}^i,\mathbb{F})$-martingale $\Psi^i(M)$ is given by

$$\Psi^i(M)_t=M_t-[\mathcal{L}(Z^{X,i})^c,M^c]_t-\sum_{0<s\le t}\frac{1}{Z_s^{X,i}}\Delta Z_s^{X,i}\Delta M_s. \tag{8.41}$$

For every $i=2,\dots,n$, the process Y^i has the following representation under $\widetilde{\mathbb{P}}^i$

$$Y_t^i=Y_0^i+\int_{(0,t]}\zeta_s^i\cdot d\widetilde{\Psi}^i(M)_s, \tag{8.42}$$

where $\zeta^i=(\zeta^{i,1},\dots,\zeta^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process and the $\mathbb{R}^k$-valued $(\widetilde{\mathbb{P}}^i,\mathbb{F})$-martingale $\widetilde{\Psi}^i(M)$ is given by

$$\widetilde{\Psi}^i(M)_t=M_t-[\mathcal{L}(Z^{Y,i})^c,M^c]_t-\sum_{0<s\le t}\frac{1}{Z_s^{Y,i}}\Delta Z_s^{Y,i}\Delta M_s. \tag{8.43}$$

The equalities $\Psi^i(M)^c=M^c$, $i=1,\dots,n$ and $\widetilde{\Psi}^i(M)^c=M^c$, $i=2,\dots,n$ are valid.

Proof. Let us fix $i=1,\dots,n$. By parts (iv) and (vi) in Assumptions 8.4.3, the process X^i is a $(\mathbb{P}^i,\mathbb{F})$-martingale. In view of part (ii) in Lemma 8.4.1, the PRP holds with respect to $\mathbb{F}$ under $\mathbb{P}^i$ with the spanning $(\mathbb{P}^i,\mathbb{F})$-martingale $\Psi^i(M)$. This yields the integral representation of X^i under $\mathbb{P}^i$. The proof of the second statement is similar, with parts (v) and (vi) in Assumptions 8.4.3 used. $\square$

Proposition 8.4.3. *For every $i=1,\dots,n$, the semimartingale decomposition of the $(\mathbb{P}^i,\mathbb{F})$-martingale $\Psi^i(M)$ under the probability measure $\mathbb{P}^n=\mathbb{P}$ is given by*

$$\begin{aligned}\Psi^i(M)_t=M_t&-\sum_{j=i+1}^n\left[\int_{(0,t]}\frac{(\zeta_s^j-\xi_s^j)\cdot d[M^c]_s}{Y_s^j-X_s^j}-\int_{(0,t]}\frac{(\xi_s^{j-1}-\zeta_s^j)\cdot d[M^c]_s}{X_s^{j-1}-Y_s^j}\right]\\&-\sum_{0<s\le t}\frac{1}{Z_s^{X,i}}\,\Delta Z_s^{X,i}\Delta M_s.\end{aligned}$$

For every $i = 2, \dots, n$, the semimartingale decomposition of the $(\widetilde{\mathbb{P}}^i, \mathbb{F})$-martingale $\widetilde{\Psi}^i(M)$ under the probability measure $\mathbb{P}^n = \mathbb{P}$ is given by

$$\begin{aligned}\widetilde{\Psi}^i(M)_t = M_t &- \int_{(0,t]} \frac{(\xi_s^{i-1} - \xi_s^i) \cdot d[M^c]_s}{X_s^{i-1} - X_s^i} + \int_{(0,t]} \frac{(\xi_s^{i-1} - \zeta_s^i) \cdot d[M^c]_s}{X_s^{i-1} - Y_s^i} \\ &- \sum_{j=i+1}^n \left[\int_{(0,t]} \frac{(\zeta_s^j - \xi_s^j) \cdot d[M^c]_s}{Y_s^j - X_s^j} - \int_{(0,t]} \frac{(\xi_s^{j-1} - \zeta_s^j) \cdot d[M^c]_s}{X_s^{j-1} - Y_s^j} \right] \\ &- \sum_{0<s\le t} \frac{1}{Z_s^{Y,i}} \Delta Z_s^{Y,i} \Delta M_s.\end{aligned}$$

Proof. Using (8.18), (8.20) and (8.38), we obtain

$$\begin{aligned}\mathcal{L}(Z^{X,i})_t^c &= \sum_{j=i+1}^n \left[\mathcal{L}(Y^j - X^j)_t^c - \mathcal{L}(X^{j-1} - Y^j)_t^c \right] \\ &= \sum_{j=i+1}^n \left[\int_{(0,t]} \frac{dY_s^{j,c} - dX_s^{j,c}}{Y_s^j - X_s^j} - \int_{(0,t]} \frac{dX_s^{j-1,c} - dY_s^{j,c}}{X_s^{j-1} - Y_s^j} \right].\end{aligned}$$

Consequently,

$$\begin{aligned}[\mathcal{L}(Z^{X,i})^c, M^c]_t = &\sum_{j=i+1}^n \int_{(0,t]} \frac{d[Y^{j,c}, M^c]_s - d[X^{j,c}, M^c]_s}{Y_s^j - X_s^j} \\ &- \sum_{j=i+1}^n \int_{(0,t]} \frac{d[X^{j-1,c}, M^c]_s - d[Y^{j,c}, M^c]_s}{X_s^{j-1} - Y_s^j}.\end{aligned}$$

Using equations (8.40) and (8.42) and the equalities $\Psi^j(M)^c = \widetilde{\Psi}^j(M)^c = M^c$, which were established in Lemma 8.4.7, we conclude that

$$[\mathcal{L}(Z^{X,i})^c, M^c]_t = \sum_{j=i+1}^n \left[\int_{(0,t]} \frac{(\zeta_s^j - \xi_s^j) \cdot d[M^c]_s}{Y_s^j - X_s^j} - \int_{(0,t]} \frac{(\xi_s^{j-1} - \zeta_s^j) \cdot d[M^c]_s}{X_s^{j-1} - Y_s^j} \right].$$

By combining the formula above with (8.41), we obtain the desired expression for the process $\Psi^i(M)$.

To establish the formula for $\widetilde{\Psi}^i(M)$, we first note that

$$\mathcal{L}(Z^{Y,i})^c = \mathcal{L}(X^{i-1} - X^i)^c - \mathcal{L}(X^{i-1} - Y^i)^c + \mathcal{L}(Z^{X,i})^c,$$

which in turns implies that

$$[\mathcal{L}(Z^{Y,i})^c, M^c] = [\mathcal{L}(X^{i-1} - X^i)^c - \mathcal{L}(X^{i-1} - Y^i)^c, M^c] + [\mathcal{L}(Z^{X,i})^c, M^c].$$

Since the decomposition of the process $[\mathcal{L}(Z^{X,i})^c, M^c]$ was already found above, it suffices to focus on the process $\mathcal{L}(X^{i-1} - X^i)^c - \mathcal{L}(X^{i-1} - Y^i)^c$. In view of (8.20), we have that

$$\mathcal{L}(X^{i-1} - X^i)_t^c - \mathcal{L}(X^{i-1} - Y^i)_t^c = \int_{(0,t]} \frac{dX_s^{i-1,c} - dX_s^{i,c}}{X_s^{i-1} - X_s^i} - \int_{(0,t]} \frac{dX_s^{i-1,c} - dY_s^{i,c}}{X_s^{i-1} - Y_s^i}.$$

By making again use of integral representations (8.40) and (8.42), we thus obtain

$$[\mathcal{L}(X^{i-1} - X^i)^c - \mathcal{L}(X^{i-1} - Y^i)^c, M^c]_t = \int_{(0,t]} \frac{(\xi_s^{i-1} - \xi_s^i) \cdot d[M^c]_s}{X_s^{i-1} - X_s^i} - \int_{(0,t]} \frac{(\xi_s^{i-1} - \zeta_s^i) \cdot d[M^c]_s}{X_s^{i-1} - Y_s^i}.$$

After combining the decomposition of $[\mathcal{L}(Z^{X,i})^c, M^c]$ and $[\mathcal{L}(X^{i-1}-X^i)^c - \mathcal{L}(X^{i-1}-Y^i)^c, M^c]$, we conclude that

$$\begin{aligned}\widetilde{\Psi}^i(M)_t = M_t &- \int_{(0,t]} \frac{(\xi_s^{i-1}-\xi_s^i)\cdot d[M^c]_s}{X_s^{i-1}-X_s^i} + \int_{(0,t]} \frac{(\xi_s^{i-1}-\zeta_s^i)\cdot d[M^c]_s}{X_s^{i-1}-Y_s^i} \\ &- \sum_{j=i+1}^{n} \left[\int_{(0,t]} \frac{(\zeta_s^j-\xi_s^j)\cdot d[M^c]_s}{Y_s^j-X_s^j} - \int_{(0,t]} \frac{(\xi_s^{j-1}-\zeta_s^j)\cdot d[M^c]_s}{X_s^{j-1}-Y_s^j} \right] \\ &- \sum_{0<s\le t} \frac{1}{Z_s^{Y,i}}\, \Delta Z_s^{Y,i} \Delta M_s,\end{aligned}$$

which is the desired formula. □

8.4.4 Setup (D)

The setup considered in this subsection is similar to Setup (C) and thus it will be presented rather succinctly. We work here under the following standing assumptions.

Assumption 8.4.4. We postulate that:
(i) the processes $X = (X^0,\dots,X^{n-2})$ and $Y = (Y^0,\dots,Y^{n-1})$ are $\mathbb{F}$-adapted,
(ii) for every $i = 0,\dots,n-1$, the process $Z^{Y,i}$, which is given by the formula (by the usual convention, $Z^{Y,0} = 1$)

$$Z_t^{Y,i} = \widetilde{c}_i \prod_{j=0}^{i-1} \frac{Y_t^j - X_t^j}{Y_t^{j+1} - X_t^j}, \tag{8.44}$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale and $\widetilde{c}_1,\dots,\widetilde{c}_{n-1}$ are constants such that $\mathbb{E}_{\mathbb{P}}(Z_0^{Y,i}) = 1$,
(iii) for every $i = 0,\dots,n-2$, the process $Z^{X,i}$, which is given by the formula

$$Z^{X,i} = c_i \frac{Y^{i+1} - Y^i}{Y^{i+1} - X^i}\, Z^{Y,i}, \tag{8.45}$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale and $c_0,\dots,c_{n-2}$ are constants such that $\mathbb{E}_{\mathbb{P}}(Z_0^{X,i}) = 1$,
(iv) for every $i = 0,\dots,n-2$, the process X^i is a $(\mathbb{P}^i,\mathbb{F})$-martingale, where the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{X,i}$,
(v) for every $i = 0,\dots,n-1$, the process Y^i is a $(\widetilde{\mathbb{P}}^i,\mathbb{F})$-martingale, where the Radon-Nikodým density of $\widetilde{\mathbb{P}}^i$ with respect to $\mathbb{P}$ equals $Z^{Y,i}$,
(vi) the PRP holds under $(\mathbb{P},\mathbb{F})$ with the spanning $(\mathbb{P},\mathbb{F})$-martingale $M = (M^1,\dots,M^k)$.

The probability measures $\mathbb{P}^i$ and $\widetilde{\mathbb{P}}^i$ are defined as in part (i) in Lemma 8.4.1, with the Radon-Nikodým density processes $Z^{X,i}$ and $Z^{Y,i}$, respectively. Hence $\widetilde{\mathbb{P}}^0 = \mathbb{P}$ since $Z^{Y,0} = 1$.

Lemma 8.4.8. *(i) For every $i = 0,\dots,n-2$, the process X^i admits the following representation under $\mathbb{P}^i$*

$$X_t^i = X_0^i + \int_{(0,t]} \xi_s^i \cdot d\Psi^i(M)_s, \tag{8.46}$$

where $\xi^i = (\xi^{i,1},\dots,\xi^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process and the $\mathbb{R}^k$-valued $(\mathbb{P}^i,\mathbb{F})$-martingale $\Psi^i(M)$ equals

$$\Psi^i(M)_t = M_t - [\mathcal{L}(Z^{X,i})^c, M^c]_t - \sum_{0<s\le t} \frac{1}{Z_s^{X,i}} \Delta Z_s^{X,i} \Delta M_s.$$

(ii) For every $i = 0,\dots,n-1$, the process Y^i has the following representation under $\widetilde{\mathbb{P}}^i$

$$Y_t^i = Y_0^i + \int_{(0,t]} \zeta_s^i \cdot d\widetilde{\Psi}^i(M)_s, \tag{8.47}$$

where $\zeta^i = (\zeta^{i,1}, \dots, \zeta^{i,k})$ *is an* $\mathbb{R}^k$*-valued,* $\mathbb{F}$*-predictable process and the* $\mathbb{R}^k$*-valued* $(\widetilde{\mathbb{P}}^i, \mathbb{F})$*-martingale* $\widetilde{\Psi}^i(M)$ *equals*

$$\widetilde{\Psi}^i(M)_t = M_t - [\mathcal{L}(Z^{Y,i})^c, M^c]_t - \sum_{0<s\le t} \frac{1}{Z_s^{Y,i}} \Delta Z_s^{Y,i} \Delta M_s.$$

The equalities $\Psi^i(M)^c = M^c$, $i = 0, \dots, n-2$ *and* $\widetilde{\Psi}^i(M)^c = M^c$, $i = 0, \dots, n-1$ *are valid.*

Proof. Let us fix $i = 0, \dots, n-2$. By parts (iv) and (vi) in Assumptions 8.4.4, the process X^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale. In view of part (ii) in Lemma 8.4.1, the PRP holds with respect to $\mathbb{F}$ under $\mathbb{P}^i$ with the spanning $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$. This proves the first part of the lemma. The proof of the second statement is similar, with parts (v) and (vi) in Assumptions 8.4.4 used. □

To derive the formulae of Proposition 8.4.4, it suffices to make a suitable change of the index and to permute the processes X^i and Y^j in the proof of Proposition 8.4.3.

Proposition 8.4.4. *For every* $i = 0, \dots, n-2$*, the semimartingale decomposition of the* $(\mathbb{P}^i, \mathbb{F})$*-martingale* $\Psi^i(M)$ *under the probability measure* $\widetilde{\mathbb{P}}^0 = \mathbb{P}$ *is given by*

$$\begin{aligned}
\Psi^i(M)_t = M_t &- \int_{(0,t]} \frac{(\zeta_s^{i+1} - \zeta_s^i) \cdot d[M^c]_s}{Y_s^{i+1} - Y_s^i} + \int_{(0,t]} \frac{(\zeta_s^{i+1} - \xi_s^i) \cdot d[M^c]_s}{Y_s^{i+1} - X_s^i} \\
&- \sum_{j=0}^{i-1} \left[\int_{(0,t]} \frac{(\zeta_s^j - \xi_s^j) \cdot d[M^c]_s}{Y_s^j - X_s^j} - \int_{(0,t]} \frac{(\zeta_s^{j+1} - \xi_s^j) \cdot d[M^c]_s}{Y_s^{j+1} - X_s^j} \right] \\
&- \sum_{0<s\le t} \frac{1}{Z_s^{X,i}} \Delta Z_s^{X,i} \Delta M_s.
\end{aligned}$$

For every $i = 0, \dots, n-1$*, the semimartingale decomposition of the* $(\widetilde{\mathbb{P}}^i, \mathbb{F})$*-martingale* $\widetilde{\Psi}^i(M)$ *under the probability measure* $\widetilde{\mathbb{P}}^0 = \mathbb{P}$ *is given by*

$$\begin{aligned}
\widetilde{\Psi}^i(M)_t = M_t &- \sum_{j=0}^{i-1} \left[\int_{(0,t]} \frac{(\zeta_s^j - \xi_s^j) \cdot d[M^c]_s}{Y_s^j - X_s^j} - \int_{(0,t]} \frac{(\zeta_s^{j+1} - \xi_s^j) \cdot d[M^c]_s}{Y_s^{j+1} - X_s^j} \right] \\
&- \sum_{0<s\le t} \frac{1}{Z_s^{Y,i}} \Delta Z_s^{Y,i} \Delta M_s.
\end{aligned}$$

8.5 Market Models for Forward CDS Spreads

We are in a position to apply the abstract setups developed in Section 8.4 to the modeling of finite families of forward CDS spreads in various configurations. As mentioned in the introduction, we will first consider the simplest case of deterministic interest rate. Subsequently, we will proceed to the study of more general, and thus more practically relevant, market models. The aim in all models is to derive the joint dynamics of a given family of forward CDS spreads under a single measure by first finding expressions for all ratios of *defaultable annuities* (annuity deflated swap numéraires) in terms of forward CDS spreads from a predetermined family. In this section, it will be assumed throughout that the σ-field $\mathcal{F}_0$ is trivial. Although this assumption is not necessary for our further developments, it is commonly used in financial modeling, since it fits well the interpretation of the σ-field $\mathcal{F}_0$ as representing the (deterministic) market data available at time 0.

8.5.1 Model of One-Period CDS Spreads

Following the papers by Brigo [22, 23] and Schlögl [101], we will first examine the issue of properties and construction of a market model for a family of one-period forward CDS spreads under the assumption that the interest rate is deterministic. It is worth noting that the situation considered in this subsection is formally similar to the modeling of a family of discrete-tenor forward LIBORs (see, for instance, Brace [18], Jamshidian [53, 54] and Musiela and Rutkowski [89, 90]). However, we need also to examine here the issue of existence of a default time consistent with a postulated dynamics of forward CDS spreads. Needless to say that this issue does not arise in the problem of modeling default-free market interest rates, such as: forward LIBORs or, more generally, forward swap rates.

Bottom-Up Approach

By the *bottom-up approach* we mean a modeling approach in which one starts with some credit risk model (where the default time τ and the associated survival process G are defined) as well as a term structure of interest rates. All other quantities of interest are subsequently computed from these fundamentals, in particular, the joint dynamics of a family of forward CDS spreads. In this sense, we are building up starting from the bottom.

We refer to Section 8.2 for the description of a generic forward CDS and the definition of the associated fair forward CDS spread. Recall that we consider a tenor structure $\mathcal{T} = \{T_0 < T_1 < \cdots < T_n\}$ with $T_0 \geq 0$ and we write $a_i = T_i - T_{i-1}$. We start by noting that, using formulae (8.9) and (8.10), one can derive the following expression for the pre-default forward CDS spread $\kappa^i := \kappa^{i-1,i}$ corresponding to the one-period forward CDS starting at T_{i-1} and maturing at T_i

$$1 + \widehat{a}_i \kappa^i_t = \frac{\mathbb{E}_{\mathbb{Q}}(\beta(t,T_i)\mathbb{1}_{\{\tau > T_{i-1}\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(\beta(t,T_i)\mathbb{1}_{\{\tau > T_i\}} \,|\, \mathcal{F}_t)}, \quad \forall\, t \in [0, T_{i-1}], \tag{8.48}$$

where we denote $\widehat{a}_i = a_i/(1-\delta_i)$ (recall that δ_i is a non-random recovery rate). In addition to the family of one-period forward CDS spreads $\kappa^1, \ldots, \kappa^n$, we also consider in Subsections 8.5.1 and 8.5.2 the family of forward LIBORs $L^1, \ldots, L^n$, which are defined by the formula

$$1 + a_i L^i_t = \frac{B(t, T_{i-1})}{B(t, T_i)}, \tag{8.49}$$

where for every $T > 0$ we denote by $B(t,T)$ the price at time t of the default-free unit zero-coupon bond maturing at time T. Throughout Subsection 8.5.1, we work under the standing assumption that the interest rate is deterministic, so that

$$1 + \widehat{a}_i \kappa^i_t = \frac{\mathbb{Q}(\tau > T_{i-1} \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_i \,|\, \mathcal{F}_t)}, \quad \forall\, t \in [0, T_{i-1}], \tag{8.50}$$

It then follows immediately from (8.3) and (8.50) that the process κ^i is positive, for $i = 1, \ldots, n$.

We observe that the right-hand side in (8.50) can be represented as follows, for every $i = 1, \ldots, n$ and $t \in [0, T_{i-1}]$,

$$1 + \widehat{a}_i \kappa^i_t = \frac{a_i \beta(t, T_i)}{a_{i-1}\beta(t, T_{i-1})} \frac{\widetilde{A}^{i-2,i-1}_t}{\widetilde{A}^{i-1,i}_t} = \frac{a_i}{a_{i-1}} \frac{1}{1 + a_i L^i_t} \frac{\widetilde{A}^{i-2,i-1}_t}{\widetilde{A}^{i-1,i}_t}. \tag{8.51}$$

Formula (8.51) shows that $1 + \widehat{a}_i \kappa^i_t$ is equal, up to a deterministic function, to the ratio of *defaultable annuities* corresponding to the forward CDS spreads κ^{i-1} and κ^i. Since the interest rates are now assumed to be deterministic, we find it easier to work directly with (8.50), however. Let us mention that the case of random interest rates is examined in Subsection 8.5.2.

Let us define the probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{T_n})$ by setting, for every $t \in [0, T_n]$,

$$\eta_t = \frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{\mathbb{Q}(\tau > T_n \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_n)}.$$

Lemma 8.5.1. *For every $i = 0, \dots, n$, the process $\bar{Z}^{\kappa,i}$, which is given by the formula*

$$\bar{Z}^{\kappa,i}_t = \prod_{j=i+1}^{n} \big(1 + \widehat{a}_j \kappa^j_t\big), \quad \forall\, t \in [0, T_i],$$

is a $(\mathbb{P}, \mathbb{F})$-martingale where, by the usual convention, we set $\bar{Z}^{\kappa,n}_t = 1$.

Proof. It suffices observe that from (8.50) we obtain

$$\bar{Z}^{\kappa,i}_t = \frac{\mathbb{Q}(\tau > T_i \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_n \,|\, \mathcal{F}_t)}. \tag{8.52}$$

It is now easily seen that the process $\bar{Z}^{\kappa,i}\eta$ is a $(\mathbb{Q}, \mathbb{F})$-martingale, which in turn implies that the asserted property is valid. □

In the next step, for any $i = 1, \dots, n$, we define the probability measure $\mathbb{P}^i$ equivalent to $\mathbb{P}$ by setting

$$\frac{d\mathbb{P}^i}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z^{\kappa,i}_t := c_i \bar{Z}^{\kappa,i}_t \simeq \prod_{j=i+1}^{n} \big(1 + \widehat{a}_j \kappa^j_t\big), \tag{8.53}$$

where $c_i = \frac{\mathbb{Q}(\tau > T_n)}{\mathbb{Q}(\tau > T)}$ is the normalizing constant and the symbol $\simeq$ means that the equality holds up to a normalizing constant. Note that the equality $Z^{\kappa,n}_t = 1$ holds and thus $\mathbb{P}^n = \mathbb{P}$.

Lemma 8.5.2. *For any fixed $i = 1, \dots, n$, the one-period forward CDS spread κ^i is a positive $(\mathbb{P}^i, \mathbb{F})$-martingale.*

Proof. It is enough to show that $1 + \widehat{a}_i \kappa^i_t$ is a $(\mathbb{P}^i, \mathbb{F})$-martingale. To this end, it suffices to combine formulae (8.50) and (8.52). □

The relationships between various martingale measures for the family of one-period forward CDS spreads are summarized in the following diagram

$$\mathbb{Q} \xrightarrow{\frac{d\mathbb{P}^n}{d\mathbb{Q}}} \mathbb{P}^n \xrightarrow{\frac{d\mathbb{P}^{n-1}}{d\mathbb{P}^n}} \mathbb{P}^{n-1} \xrightarrow{\frac{d\mathbb{P}^{n-2}}{d\mathbb{P}^{n-1}}} \dots \xrightarrow{\frac{d\mathbb{P}^2}{d\mathbb{P}^3}} \mathbb{P}^2 \xrightarrow{\frac{d\mathbb{P}^1}{d\mathbb{P}^2}} \mathbb{P}^1$$

where

$$\frac{d\mathbb{P}^n}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{\mathbb{Q}(\tau > T_n \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_n)},$$
$$\frac{d\mathbb{P}^{i-1}}{d\mathbb{P}^i}\Big|_{\mathcal{F}_t} = \widehat{c}_i (1 + \widehat{a}_i \kappa^i_t) \simeq \frac{\mathbb{Q}(\tau > T_{i-1} \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_i \,|\, \mathcal{F}_t)},$$

where $\widehat{c}_i = \frac{\mathbb{Q}(\tau > T_i)}{\mathbb{Q}(\tau > T_{i-1})}$ is the normalizing constant.

Let us assume, in addition, that the PRP holds under $\mathbb{P} = \mathbb{P}^n$ with the spanning $(\mathbb{P}, \mathbb{F})$-martingale M. Then this property is also valid with respect to the filtration $\mathbb{F}$ under any probability measure $\mathbb{P}^i$ for $i = 1, \dots, n$ with the spanning $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$. Therefore, for every $i = 1, \dots, n$, the positive $(\mathbb{P}^i, \mathbb{F})$-martingale κ^i admits the following integral representation under $\mathbb{P}^i$

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-} \sigma^i_s \cdot d\Psi^i(M)_s, \tag{8.54}$$

where σ^i is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process. For the sake of convenience, we will henceforth refer to σ^i as the 'volatility' of the forward CDS spread κ^i.

To sum up, we have just shown the existence of the volatility processes for one-period forward CDS spreads. It will be now not difficult to derive the joint dynamics of these processes under a single probability measure, for instance, the martingale measure $\mathbb{P} = \mathbb{P}^n$. However, since the joint dynamics of the one-period CDS spreads are the same as in the top-down approach presented in the foregoing subsection, we do not provide the explicit formula here and we refer instead the reader to equation (8.60).

Top-Down Approach

As was explained above, in the standard bottom-up approach to modeling of CDS spreads, we start by choosing a term structure model and we combine it with the specification of the default time. Together with the assumption that the PRP holds, we can subsequently derive the joint dynamics and show the existence of a family of volatility processes.

By contrast, in the *top-down approach* to a market model, we take as inputs a family of volatility processes and a family of driving martingales. We then derive the joint dynamics of a given family of forward CDS spreads under a single probability measure and show that they are uniquely specified by the choice of the volatility processes and driving martingales. Thus, instead of doing computation from a given credit and term structure model, one can directly specify the joint dynamics of a family of CDS spreads through the choice of the volatility processes and driving martingales. This justifies the name 'top-down' approach. In principle, it is also possible to construct the corresponding default time, but this issue is not dealt with in this chapter.

In this subsection, we will show how to apply the top-down approach in order to construct the model of one-period forward CDS spreads under the assumption that the interest rate is deterministic. To achieve this goal, we will work under the following standing assumptions, which are motivated by the analysis of the bottom-up approach.

Assumption 8.5.1. We are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and we postulate that:
(i) the initial values of processes $\kappa^1, \ldots, \kappa^n$ are given,
(ii) for every $i = 1, \ldots, n$, the process $Z^{\kappa,i}$, which is given by the formula (by the usual convention, $Z^{\kappa,n} = 1$)

$$Z_t^{\kappa,i} = c_i \prod_{j=i+1}^{n} (1 + \widehat{a}_j \kappa_t^j), \tag{8.55}$$

is a positive $(\mathbb{P}, \mathbb{F})$-martingale, where c_i is a constant such that $Z_0^{\kappa,i} = 1$,
(iii) for every $i = 1, \ldots, n$, the process κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale, where the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{\kappa,i}$, so that, in particular, $\mathbb{P}^n = \mathbb{P}$,
(iv) for every $i = 1, \ldots, n$, the process κ^i has the following representation under $(\mathbb{P}^i, \mathbb{F})$

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \kappa_{s-}^i \widetilde{\sigma}_s^i \cdot d\widetilde{N}_s^i, \tag{8.56}$$

where $\widetilde{N}^i$ is an $\mathbb{R}^{k_i}$-valued $(\mathbb{P}^i, \mathbb{F})$-martingale and $\widetilde{\sigma}^i$ is an $\mathbb{R}^{k_i}$-valued, $\mathbb{F}$-predictable, $\widetilde{N}^i$-integrable process.

In view of part (iii) in Assumptions 8.5.1, the following definition is rather obvious.

Definition 8.5.1. The probability measure $\mathbb{P}^i$ is called the *martingale measure* for the one-period forward CDS spread κ^i.

We should make some important comments here. According to the top-down approach, the processes $\kappa^1, \ldots, \kappa^n$ are in fact not given in advance, but should be constructed. Also, the default time is not defined, so the forward CDS spreads cannot be computed using formulae (8.9), (8.10) and (8.11). Therefore, the volatility processes $\widetilde{\sigma}^1, \ldots, \widetilde{\sigma}^n$ appearing in formula (8.56), as well as the driving martingales $\widetilde{N}^1, \ldots, \widetilde{N}^n$, can be chosen arbitrarily by the modeler. Put another way, the volatility processes and driving martingales should now be seen as model inputs.

As an output in the top-down construction of a market model, we obtain the joint dynamics of the forward CDS spreads $\kappa^1, \ldots, \kappa^n$ under a common probability measure. The form of the joint dynamics obtained below should be seen as a necessary condition for the arbitrage-free property of the model. Unfortunately, sufficient conditions for the arbitrage-free property are more difficult to handle.

Remark 8.5.1. It is worth mentioning that assumption (8.56) can be replaced by the following one

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \widetilde{\sigma}_s^i \cdot d\widetilde{N}_s^i,$$

which, in fact, is more general than (8.56). All foregoing results can be easily adjusted to this alternative postulate. Similar comments apply to other variants of market models of forward CDS spreads (and forward LIBORs), which are examined in the foregoing subsections, so they will not be repeated in what follows.

Lemma 8.5.3. *Let $k = k_1 + \cdots + k_n$. There exists an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale M such that the process κ^i admits the following representation under the martingale measure $\mathbb{P}^i$*

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \kappa_{s-}^i \sigma_s^i \cdot d\Psi^i(M)_s, \tag{8.57}$$

where σ^i is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process and the process $\Psi^i(M)$ is given by (8.24).

Proof. Using the bijectivity of the Girsanov transform, for every $i = 1, \ldots, n$ one can establish the existence of an $\mathbb{R}^{k_i}$-valued $(\mathbb{P}, \mathbb{F})$-martingale N^i such that $\widetilde{N}^i = \Psi^i(N^i) = \Psi_{Z^{\kappa,i}}(N^i)$. Hence (8.56) yields

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \kappa_{s-}^i \widetilde{\sigma}_s^i \cdot d\Psi^i(N^i)_s.$$

To complete the proof, it suffices to define the $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale M by setting $M = (N^1, \ldots, N^n)$ and to formally extend each process $\mathbb{R}^{k_i}$-valued, $\mathbb{F}$-predictable process $\widetilde{\sigma}^i$ to the corresponding $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process σ^i by setting to zero all irrelevant components of σ^i. □

It is worth noting that $\widetilde{N}^n = N^n$ since $Z^{\kappa,n} = 1$. Consequently, for $i = n$ equation (8.57) becomes

$$\kappa_t^n = \kappa_0^n + \int_{(0,t]} \kappa_{s-}^n \sigma_s^n \cdot dM_s.$$

Of course, this is consistent with the interpretation of $\mathbb{P}^n$ as the martingale measure for κ^n. By contrast, for $i = 1, \ldots, n-1$ it is expected that the dynamics (8.57) of κ^i under $\mathbb{P}^n$ will also have a non-zero 'drift' term, which is due to appear, since the process $\widetilde{N}^i$ is not a $(\mathbb{P}^n, \mathbb{F})$-martingale, in general. The main result of this subsection, Proposition 8.5.1, furnishes an explicit expression for the drift term in the dynamics (8.57) of κ^i under $\mathbb{P}^n$.

Remark 8.5.2. In practice, for instance, when the processes $\widetilde{N}^i$ are Brownian motions under the respective martingale measures $\mathbb{P}^i$, the actual dimension of the underlying driving $(\mathbb{P}, \mathbb{F})$-martingale can typically be chosen to be less than $k_1 + \cdots + k_n$. For more details on explicit constructions of market models for forward CDS spreads driven by correlated Brownian motions, we refer to papers by Brigo [23] and Rutkowski [100], where the common volatilities-correlations approach is presented.

In view of Lemma 8.5.3, we may reformulate Assumptions 8.5.1 in the simpler, and more practically appealing, manner.

Assumption 8.5.2. We are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and we postulate that:
(i) the initial values of processes $\kappa^1, \ldots, \kappa^n$ are given,
(ii) for every $i = 1, \ldots, n$, the process $Z^{\kappa,i}$, which is given by the formula (by the usual convention, $Z^{\kappa,n} = 1$)

$$Z_t^{\kappa,i} = c_i \prod_{j=i+1}^{n} (1 + \widehat{a}_j \kappa_t^j), \tag{8.58}$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale, where c_i is a constant such that $Z^{\kappa,i}_0 = 1$,
(iii) for every $i = 1,\dots,n$, the process κ^i has the following representation under $(\mathbb{P},\mathbb{F})$

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-}\sigma^i_s \cdot d\Psi^i(M)_s, \tag{8.59}$$

where M is an $\mathbb{R}^k$-valued $(\mathbb{P},\mathbb{F})$-martingale, σ^i is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process and the process $\Psi^i(M)$ is given by formula (8.24) with $Z^i = Z^{\kappa,i}$.

By comparing Assumptions 8.5.2 with Assumptions 8.4.1 and Lemma 8.4.3, we see that the joint dynamics of $\kappa^1,\dots,\kappa^n$ under $\mathbb{P} = \mathbb{P}^n$ can now be computed by applying results obtained for Setup (A). Of course, in order to proceed, we need to make an implicit assumption that all relevant processes belong to the space $\mathcal{S}_1$ introduced in Section 8.3. The following lemma is a rather straightforward consequence of Proposition 8.4.1. Recall that for the $\mathbb{R}^k$-valued $(\mathbb{P},\mathbb{F})$-martingale M we have that

$$M^c = (M^1,\dots,M^k)^c = (M^{1,c},\dots,M^{k,c})$$

and the matrix $[M^c]$ is given by formula (8.25).

Lemma 8.5.4. *Under Assumptions 8.5.2, for every $i = 1,\dots,n$ the semimartingale decomposition of the $(\mathbb{P}^i,\mathbb{F})$-martingale $\Psi^i(M)$ under the martingale measure $\mathbb{P}^n = \mathbb{P}$ is given by*

$$\Psi^i(M)_t = M_t - \sum_{j=i+1}^n \int_{(0,t]} \frac{\widehat{a}_j\kappa^j_s\sigma^j_s\cdot d[M^c]_s}{1+\widehat{a}_j\kappa^j_s} - \sum_{0<s\le t}\frac{1}{Z^{\kappa,i}_s}\,\Delta Z^{\kappa,i}_s\Delta M_s,$$

where $\sigma^j = (\sigma^{j,1},\dots,\sigma^{j,k})$. More explicitly, for every $i = 1,\dots,n$ and $l = 1,\dots,k$,

$$\Psi^i(M^l)_t = M^l_t - \sum_{j=i+1}^n \int_{(0,t]} \frac{\widehat{a}_j\kappa^j_s}{1+\widehat{a}_j\kappa^j_s}\sum_{m=1}^k \sigma^{j,m}_s\,d[M^{l,c},M^{m,c}]_s - \sum_{0<s\le t}\frac{1}{Z^{\kappa,i}_s}\,\Delta Z^{\kappa,i}_s\Delta M^l_s.$$

The next result, which follows easily by combining formula (8.57) with Lemma 8.5.4, furnishes an explicit expression for the joint dynamics of the family of one-period forward CDS spreads under deterministic interest rates. An important conclusion from Proposition 8.5.1 is that the joint dynamics of one-period forward CDS spreads are uniquely specified, under Assumptions 8.5.1, by the choice of the driving $(\mathbb{P}^i,\mathbb{F})$-martingales $\widetilde{N}^1,\dots,\widetilde{N}^n$ and the volatility processes $\widetilde{\sigma}^1,\dots,\widetilde{\sigma}^n$ or, more practically, under Assumptions 8.5.2, by the choice of the driving $(\mathbb{P},\mathbb{F})$-martingale M under $\mathbb{P}$ and the volatility processes $\sigma^1,\dots,\sigma^n$.

Proposition 8.5.1. *Under Assumptions 8.5.2, the joint dynamics of forward CDS spreads $\kappa^1,\dots,\kappa^n$ under the martingale measure $\mathbb{P} = \mathbb{P}^n$ are given by the following n-dimensional stochastic differential equation driven by the $(\mathbb{P},\mathbb{F})$-martingale M: for every $i = 1,\dots,n$ and $t \in [0,T_{i-1}]$*

$$\begin{aligned} d\kappa^i_t = {} & \sum_{l=1}^k \kappa^i_{t-}\sigma^{i,l}_t\,dM^l_t - \sum_{j=i+1}^n \frac{\widehat{a}_j\kappa^i_t\kappa^j_t}{1+\widehat{a}_j\kappa^j_t}\sum_{l,m=1}^k \sigma^{i,l}_t\sigma^{j,m}_t\,d[M^{l,c},M^{m,c}]_t \\ & - \frac{\kappa^i_{t-}}{Z^{\kappa,i}_t}\,\Delta Z^{\kappa,i}_t\sum_{l=1}^k \sigma^{i,l}_t\Delta M^l_t. \end{aligned} \tag{8.60}$$

We need also to address some consistency issues regarding the top-down methodology. In the top-down approach, we start with a given family of volatility processes related to some driving martingale and we derive the SDE (8.60). Therefore, it is necessary to establish the existence and uniqueness result for a solution to the SDE (8.60) governing the forward CDS spreads κ^i, $i = 1,\dots,n$ under $\mathbb{P}$. Moreover, we need to show that given a solution to (8.60) exists, the processes $Z^{\kappa,i}$ computed from the SDE (8.60) is indeed a $(\mathbb{P},\mathbb{F})$-martingale, as it was assumed.

Proposition 8.5.2. *Let σ^i, $i = 1, \dots, n$ be $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable processes. Assume that the joint dynamics of processes κ^i, $i = 1, \dots, n$ under $\mathbb{P}$ are given by (8.60) with $Z^{\kappa,i}$ given by (8.55). Then:*
(i) the processes κ^i, $i = 1, \dots, n$ are well defined, specifically, the stochastic differential equation (8.60) admits a unique strong solution $(\kappa^1, \dots, \kappa^n)$,
(ii) for every $i = 1, \dots, n$, the process $Z^{\kappa,i}$ given by (8.55) is a $(\mathbb{P}, \mathbb{F})$-martingale,
(iii) for every $i = 1, \dots, n$, the process κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale governed by (8.54), where in turn the probability measure $\mathbb{P}^i$ is given by (8.53).

Proof. Part (i) is rather standard; it suffices to employ the classic result on stochastic differential equations with locally Lipschitz continuous coefficients. Part (ii) can be established by induction. It suffices to focus on processes $Z^{\kappa,i}$ given by (8.55). From (8.60), we deduce that κ^n is a $(\mathbb{P}, \mathbb{F})$-martingale and thus the process $Z^{\kappa,n-1}_t = \big(1 + \widehat{a}_n \kappa^n_t\big)$ is a $(\mathbb{P}, \mathbb{F})$-martingale as well. Assume now that $Z^{\kappa,i+1}$ is a $(\mathbb{P}, \mathbb{F})$-martingale so that, in view of (8.55) and (8.60),

$$dZ^{\kappa,i+1}_t = \sum_{j=i+2}^{n} \frac{Z^{\kappa,i+1}_t}{\big(1 + \widehat{a}_j \kappa^j_t\big)} \, \kappa^j_t \sigma^j_t \cdot dM_t. \tag{8.61}$$

Note that

$$Z^{\kappa,i}_t = \big(1 + \widehat{a}_{i+1} \kappa^{i+1}_t\big) Z^{\kappa,i+1}_t. \tag{8.62}$$

To show that $Z^{\kappa,i}$ is a $(\mathbb{P}, \mathbb{F})$-martingale, it suffices to check that the drift term in the dynamics of this process vanishes. For this purpose, we apply the Itô formula to the right-hand side in (8.55) and we employ formulae (8.60) and (8.61) in order to compute the covariation $[\kappa^{i+1}, Z^{\kappa,i+1}]$. Part (iii) is an immediate consequence of the postulated dynamics (8.60) and Lemma 8.5.4. □

8.5.2 Model of LIBORs and One-Period CDS Spreads

In this subsection, we relax the assumption that interest rate is deterministic. Our goal is to examine the joint dynamics of one-period forward CDS spreads $\kappa^1, \dots, \kappa^n$ and forward LIBORs $L^1, \dots, L^n$ associated with the tenor structure $\mathcal{T} = \{T_0 < T_1 < \cdots < T_n\}$. In general, this task seems to be quite difficult and thus we will examine the special case when an independence property holds.

Bottom-Up Approach

In the bottom-up approach, we will work under the assumption that the term structure model and the default intensity are independent under the risk-neutral probability measure, denoted as $\mathbb{Q}$ in what follows. For a fixed time horizon $T = T_n$, we work under the following assumptions.

Assumption 8.5.3. We are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$ and we postulate that:
(i) the risk-neutral default intensity process $(\lambda_t)_{t \in [0,T]}$ generates the filtration $\mathbb{F}^\lambda = (\mathcal{F}^\lambda_t)_{t \in [0,T]}$,
(ii) the interest rate process $(r_t)_{t \in [0,T]}$ generates the filtration $\mathbb{F}^r = (\mathcal{F}^r_t)_{t \in [0,T]}$,
(iii) the interest rate r and the default intensity λ are independent processes under $\mathbb{Q}$,
(iv) the (H)-*hypothesis* holds, meaning that the filtration $\mathbb{F} := \mathbb{F}^r \vee \mathbb{F}^\lambda$ is *immersed* in $\mathbb{G}$.

For the (H)-*hypothesis* (also known as the *immersion property* or the *martingale invariance property*) we refer, for instance, to Section 6.1 in [12] or Section 3.2 in [14]. Recall that, under the (H)-hypothesis, the risk-neutral default intensity λ is an $\mathbb{F}$-predictable process such that the survival process G admits the following representation, for every $t \in [0, T]$,

$$G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = \exp\bigg(- \int_0^t \lambda_u \, du \bigg).$$

It is thus easy to deduce from Assumptions 8.5.3 that the interest rate r and the survival process G are independent under the risk-neutral probability measure $\mathbb{Q}$. In the bottom-up approach, our goal is to derive the joint dynamics of the family of one-period forward CDS spreads $\kappa^1, \ldots, \kappa^n$, where (cf. (8.50))

$$1 + \widehat{a}_i \kappa^i_t = \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t\right)}, \tag{8.63}$$

and the family of forward LIBORs $L^1, \ldots, L^n$, which are given by (cf. (8.49))

$$1 + a_i L^i_t = \frac{B(t,T_{i-1})}{B(t,T_i)}, \tag{8.64}$$

where $B(t,T)$ is the price at time t of the default-free, unit zero-coupon bond maturing at time T. We will now attempt to simplify the formula for the forward CDS spread κ^i.

Lemma 8.5.5. *Let Assumptions 8.5.3 be satisfied. Then for every $i = 1, \ldots, n$, the CDS spread κ^i has the following representation, for every $t \in [0, T_{i-1}]$,*

$$1 + \widehat{a}_i \kappa^i_t = \frac{\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}_t\right)}. \tag{8.65}$$

Proof. Using (8.63) and the properties of conditional expectation, we obtain

$$1 + \widehat{a}_i \kappa^i_t = \frac{\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_{T_i}\right) \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_{T_i}\right) \,\middle|\, \mathcal{F}_t\right)} = \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_{T_i}\right) \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}_{T_i}\right) \,\middle|\, \mathcal{F}_t\right)}.$$

In view of the immersion property postulated in part (iv) of Assumptions 8.5.3, we have that

$$\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_{T_i}\right) = \mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_{T_{i-1}}\right) = G_{T_{i-1}},$$

and thus

$$1 + \widehat{a}_i \kappa^i_t = \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_{T_{i-1}}\right) \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}_{T_i}\right) \,\middle|\, \mathcal{F}_t\right)} = \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)G_{T_{i-1}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)G_{T_i} \,\middle|\, \mathcal{F}_t\right)}.$$

Since the interest rate and default intensity are assumed to be independent under $\mathbb{Q}$, using Lemma 8.4.4, we obtain the following chain of equalities

$$\begin{aligned}
1 + \widehat{a}_i \kappa^i_t &= \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)G_{T_{i-1}} \,\middle|\, \mathcal{F}^r_t \vee \mathcal{F}^\lambda_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)G_{T_i} \,\middle|\, \mathcal{F}^r_t \vee \mathcal{F}^\lambda_t\right)} \\
&= \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i) \,\middle|\, \mathcal{F}^r_t\right)\mathbb{E}_{\mathbb{Q}}\left(G_{T_{i-1}} \,\middle|\, \mathcal{F}^\lambda_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i) \,\middle|\, \mathcal{F}^r_t\right)\mathbb{E}_{\mathbb{Q}}\left(G_{T_i} \,\middle|\, \mathcal{F}^\lambda_t\right)} \\
&= \frac{\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}^\lambda_t\right)}{\mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}^\lambda_t\right)} = \frac{\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}_t\right)}.
\end{aligned}$$

This shows that equality (8.65) is valid. □

From (8.65), we see that, under Assumptions 8.5.3, we once again obtain expression (8.50), as in the case of deterministic interest rates. Let us point out that it is sufficient to start from (8.65) and to follow the exact derivation for the case of deterministic interest rate, in order to obtain the joint dynamics of the family of one-period forward CDS spreads. We wish, however, to obtain a model in which the joint dynamics of forward LIBORs and one-period forward CDS spreads are specified.

To this end, we first define the probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{T_n})$ by postulating that the Radon-Nikodým density is given by

$$\eta_t = \frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_n)\mathbb{1}_{\{\tau > T_n\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}(\beta(t,T_n)\mathbb{1}_{\{\tau > T_n\}})}. \tag{8.66}$$

For every $i, j = 1, \dots, n$, we will now derive expressions for the ratios of defaultable annuities (i.e., *annuity deflated swap numéraires*)

$$\frac{\widetilde{A}_t^{i-1,i}}{\widetilde{A}_t^{j-1,j}} = \frac{a_i}{a_j} \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_j)\mathbb{1}_{\{\tau > T_j\}} \,\middle|\, \mathcal{F}_t\right)} \tag{8.67}$$

in terms of one-period CDS spreads and forward LIBORs.

Lemma 8.5.6. *Under Assumptions 8.5.3, the annuity deflated swap numéraire satisfies, for every $i = 2, \dots, n$,*

$$\frac{\widetilde{A}_t^{i-2,i-1}}{\widetilde{A}_t^{i-1,i}} = \frac{a_{i-1}}{a_i}(1 + \widehat{a}_i \kappa_t^i)(1 + a_i L_t^i). \tag{8.68}$$

Proof. In view of (8.9) and the assumed independence of the interest rate and the default intensity, we obtain

$$\frac{\widetilde{A}_t^{i-2,i-1}}{\widetilde{A}_t^{i-1,i}} = \frac{a_{i-1}}{a_i} \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_{i-1})\mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t\right)} = \frac{a_{i-1}}{a_i} \frac{B(t,T_{i-1})\mathbb{Q}\left(\tau > T_{i-1} \,\middle|\, \mathcal{F}_t\right)}{B(t,T_i)\mathbb{Q}\left(\tau > T_i \,\middle|\, \mathcal{F}_t\right)}.$$

The result now follows by substituting (8.64) and (8.65) into the formula above. □

Formula (8.68) furnishes a collection of positive processes, which will be used to define a family of equivalent probability measures $\mathbb{P}^i$ for $i = 1, \dots, n$ such that the processes κ^i and L^i are $(\mathbb{P}^i, \mathbb{F})$-martingales. The relationships between these probability measures are represented by the following diagram

$$\mathbb{Q} \xrightarrow{\frac{d\mathbb{P}^n}{d\mathbb{Q}}} \mathbb{P}^n \xrightarrow{\frac{d\mathbb{P}^{n-1}}{d\mathbb{P}^n}} \mathbb{P}^{n-1} \xrightarrow{\frac{d\mathbb{P}^{n-2}}{d\mathbb{P}^{n-1}}} \dots \xrightarrow{\frac{d\mathbb{P}^2}{d\mathbb{P}^3}} \mathbb{P}^2 \xrightarrow{\frac{d\mathbb{P}^1}{d\mathbb{P}^2}} \mathbb{P}^1$$

where we set (recall that $\simeq$ means that the equality holds up to a normalizing constant)

$$\frac{d\mathbb{P}^n}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} = \eta_t,$$
$$\frac{d\mathbb{P}^{i-1}}{d\mathbb{P}^i}\bigg|_{\mathcal{F}_t} :\simeq \frac{\widetilde{A}_t^{i-2,i-1}}{\widetilde{A}_t^{i-1,i}} = \frac{a_{i-1}}{a_i}(1 + \widehat{a}_i \kappa_t^i)(1 + a_i L_t^i).$$

In view of (8.68), it is then not hard to see that

$$\frac{d\mathbb{P}^i}{d\mathbb{P}^n}\bigg|_{\mathcal{F}_t} := Z_t^{\kappa,L,i} :\simeq \frac{\widetilde{A}_t^{i-1,i}}{\widetilde{A}_t^{n-1,n}} = \frac{a_i}{a_n} \prod_{j=i+1}^{n} (1 + \widehat{a}_j \kappa_t^j)(1 + a_j L_t^j). \tag{8.69}$$

We define the equivalent probability measure $\mathbb{P}^i$ by postulating that the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{\kappa,L,i}$. In particular, we have that $Z_t^{\kappa,L,n} = 1$ and thus the equality $\mathbb{P}^n = \mathbb{P}$ holds. The next result shows that $\mathbb{P}^i$ is a martingale measure for the processes κ^i and L^i.

Lemma 8.5.7. *For every $i = 1, \dots, n$, the forward CDS spread κ^i and the forward LIBOR L^i are $(\mathbb{P}^i, \mathbb{F})$-martingales.*

Proof. To demonstrate that the forward CDS spread κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale, it is enough to use (8.63) and (8.69) in order to establish that the product $Z^{\kappa,L,i}(1 + \widehat{a}_i \kappa^i)$ is a $(\mathbb{P}, \mathbb{F})$-martingale. Similarly, to prove that the forward LIBOR L^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale, it suffices to use (8.64), (8.66) and (8.69) to show that, under the independence assumption, the process $\eta Z^{\kappa,L,i}(1 + a_i L^i)$ is a $(\mathbb{Q}, \mathbb{F})$-martingale. It then follows immediately that $Z^{\kappa,L,i}(1 + a_i L^i)$ is a $(\mathbb{P}, \mathbb{F})$-martingale, which in turn implies that $1 + a_i L^i$ (and thus also the forward LIBOR L^i) is a $(\mathbb{P}^i, \mathbb{F})$-martingale. □

Proposition 8.5.3. *Assume that the PRP holds with respect to* $(\mathbb{P},\mathbb{F})$ *with the spanning* $(\mathbb{P},\mathbb{F})$*-martingale* M*. Then, for every* $i=1,\dots,n$*, the positive* $(\mathbb{P}^i,\mathbb{F})$*-martingale* κ^i *has the following representation under* $\mathbb{P}^i$

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-}\sigma^i_s \cdot d\Psi^i(M)_s, \tag{8.70}$$

where σ^i *is an* $\mathbb{R}^k$*-valued,* $\mathbb{F}$*-predictable process. Similarly, for every* $i=1,\dots,n$*, the positive* $(\mathbb{P}^i,\mathbb{F})$*-martingale* L^i *admits the following representation under* $\mathbb{P}^i$

$$L^i_t = L^i_0 + \int_{(0,t]} L^i_{s-}\xi^i_s \cdot d\Psi^i(M)_s, \tag{8.71}$$

where ξ^i *is an* $\mathbb{R}^k$*-valued,* $\mathbb{F}$*-predictable process.*

The final step in the bottom-up approach is the derivation of the joint dynamics of processes $\kappa^1,\dots,\kappa^n$ and $L^1,\dots,L^n$ under $\mathbb{P}=\mathbb{P}^n$. Since the expression for these dynamics is exactly the same as in the top-down approach, we refer to Proposition 8.5.4 below.

Top-Down Approach

To construct the market model for forward LIBORs and one-period forward CDS spreads through the top-down approach, we make the following standing assumptions, which are motivated by the analysis of the bottom-up approach from the preceding section. It is worth stressing that parts (ii), (iii) and (iv) are motivated by the results obtained within the bottom-up approach, and thus they should be seen as necessary conditions for the arbitrage-free property of the market model that we are going to construct using the top-down approach.

Assumption 8.5.4. We are given a filtered probability space $(\Omega,\mathcal{G},\mathbb{F},\mathbb{P})$ and we postulate that:
(i) the initial values of processes $\kappa^1,\dots,\kappa^n$ and $L^1,\dots,L^n$ are given,
(ii) for every $i=1,\dots,n$, the following process are positive $(\mathbb{P},\mathbb{F})$-martingales

$$\prod_{j=i+1}^{n}(1+\widehat{a}_j\kappa^j_t), \quad \prod_{j=i+1}^{n}(1+a_jL^j_t),$$

(iii) for every $i=1,\dots,n$, the process $Z^{\kappa,L,i}$, which is given by (by convention, $Z^{\kappa,L,n}=1$)

$$Z^{\kappa,L,i}_t = c_i\prod_{j=i+1}^{n}(1+\widehat{a}_j\kappa^j_t)(1+a_jL^j_t),$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale, where c_i is a constant such that $Z^{\kappa,L,i}_0=1$,
(iv) for every $i=1,\dots,n$, the processes κ^i and L^i are $(\mathbb{P}^i,\mathbb{F})$-martingales, where the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{\kappa,L,i}$, so that, in particular, $\mathbb{P}^n=\mathbb{P}$,
(v) for every $i=1,\dots,n$, the process L^i satisfies

$$L^i_t = L^i_0 + \int_{(0,t]} L^i_{s-}\widehat{\xi}^i_s \cdot d\widehat{N}^i_s, \tag{8.72}$$

where $\widehat{N}^i$ is an $\mathbb{R}^{l_i}$-valued $(\mathbb{P}^i,\mathbb{F})$-martingale and $\widehat{\xi}^i$ is an $\mathbb{R}^{l_i}$-valued, $\mathbb{F}$-predictable, $\widehat{N}^i$-integrable process,
(vi) for every $i=1,\dots,n$, the process κ^i satisfies

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-}\widetilde{\sigma}^i_s \cdot d\widetilde{N}^i_s, \tag{8.73}$$

where $\widetilde{N}^i$ is an $\mathbb{R}^{k_i}$-valued $(\mathbb{P}^i,\mathbb{F})$-martingale and $\widetilde{\sigma}^i$ is an $\mathbb{R}^{k_i}$-valued, $\mathbb{F}$-predictable, $\widetilde{N}^i$-integrable process.

The volatility processes $\widehat{\xi}^i$ and $\widetilde{\sigma}^i$ should be considered as model inputs, so that they can be chosen arbitrarily. We only need to ensure that the stochastic integrals in (8.72) and (8.73) are well defined, so that the unique solutions to these stochastic differential equations are given by stochastic exponentials. The following definition is also clear, in view of part (iv) in Assumptions 8.5.4.

Definition 8.5.2. The probability measure $\mathbb{P}^i$ is called the *martingale measure* for the forward LIBOR L^i and the one-period forward CDS spread κ^i.

Let $k = k_1 + \cdots + k_n + l_1 + \cdots + l_n$. Arguing as in the proof of Lemma 8.5.3, one can show that there exists an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale M such that for every $i = 1, \ldots, n$

$$L_t^i = L_0^i + \int_{(0,t]} L_{s-}^i \xi_s^i \cdot d\Psi^i(M)_s \tag{8.74}$$

and

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \kappa_{s-}^i \sigma_s^i \cdot d\Psi^i(M)_s, \tag{8.75}$$

where $\xi^i = (\xi^{i,1}, \ldots, \xi^{i,k})$ and $\sigma^i = (\sigma^{i,1}, \ldots, \sigma^{i,k})$ are $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable processes, which extend $\widehat{\xi}^i$ and $\widetilde{\sigma}^i$, respectively.

Since $Z^{\kappa,L,n} = 1$ so that $\Psi^n(M) = M$, we deduce from (8.74) and (8.75) that under $\mathbb{P}^n = \mathbb{P}$ the $(\mathbb{P}^n, \mathbb{F})$-martingales L^n and κ^n satisfy

$$L_t^n = L_0^n + \int_{(0,t]} L_{s-}^n \xi_s^n \cdot dM_s$$

and

$$\kappa_t^n = \kappa_0^n + \int_{(0,t]} \kappa_{s-}^n \sigma_s^n \cdot dM_s.$$

By combining Proposition 8.4.2 obtained for Setup (B) with formulae (8.74) and (8.75) and parts (i)–(iii) in Assumptions 8.5.4, we obtain the following result, which yields the joint dynamics of forward LIBORs and one-period forward CDS spreads under the martingale measure $\mathbb{P}^n$ for L^n and κ^n. Hence this result uniquely specifies the desired market model up to a choice of the driving $(\mathbb{P}, \mathbb{F})$-martingale M and the volatility processes $\xi^1, \ldots, \xi^n, \sigma^1, \ldots, \sigma^n$.

Proposition 8.5.4. *Under Assumptions 8.5.4, the joint dynamics of forward LIBORs $L^1, \ldots, L^n$ and forward CDS spreads $\kappa^1, \ldots, \kappa^n$ under the martingale measure $\mathbb{P} = \mathbb{P}^n$ are given by the following $2n$-dimensional stochastic differential equation driven by the $(\mathbb{P}, \mathbb{F})$-martingale M: for every $i = 1, \ldots, n$ and $t \in [0, T_{i-1}]$*

$$\begin{aligned} dL_t^i &= \sum_{l=1}^k L_{t-}^i \xi_t^{i,l}\, dM_t^l - \sum_{j=i+1}^n \frac{a_j L_t^i L_t^j}{1 + a_j L_t^j} \sum_{l,m=1}^k \xi_t^{i,l} \xi_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\ &\quad - \sum_{j=i+1}^n \frac{\widehat{a}_j L_t^i \kappa_t^j}{1 + \widehat{a}_j \kappa_t^j} \sum_{l,m=1}^k \xi_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t - \frac{L_{t-}^i}{Z_t^{\kappa,L,i}} \Delta Z_t^{\kappa,L,i} \sum_{l=1}^k \xi_t^{i,l} \Delta M_t^l \end{aligned}$$

and for every $i = 1, \ldots, n$ and $t \in [0, T_{i-1}]$

$$\begin{aligned} d\kappa_t^i &= \sum_{l=1}^k \kappa_{t-}^i \sigma_t^{i,l}\, dM_t^l - \sum_{j=i+1}^n \frac{a_j \kappa_t^i L_t^j}{1 + a_j L_t^j} \sum_{l,m=1}^k \sigma_t^{i,l} \xi_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\ &\quad - \sum_{j=i+1}^n \frac{\widehat{a}_j \kappa_t^i \kappa_t^j}{1 + \widehat{a}_j \kappa_t^j} \sum_{l,m=1}^k \sigma_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t - \frac{\kappa_{t-}^i}{Z_t^{\kappa,L,i}} \Delta Z_t^{\kappa,L,i} \sum_{l=1}^k \sigma_t^{i,l} \Delta M_t^l. \end{aligned}$$

The uniqueness of the solution to the system of SDEs derived in Proposition 8.5.4 follows from the classic result on SDEs with locally Lipschitz continuous coefficients.

8.5.3 Model of One- and Two-Period CDS Spreads

In this subsection, we examine the family consisting of all one- and two-period CDS spreads associated with a given tenor structure $\mathcal{T} = \{T_0 < T_1 < \cdots < T_n\}$. This particular choice of a family of forward CDS spreads originates from Brigo [22, 23], who examined this setup under the assumption that the market model is driven by a multi-dimensional Brownian motion.

Bottom-Up Approach

In this subsection, we find it convenient to adopt the notation in which the forward CDS spreads are indexed by their maturity date. Specifically, the one-period forward CDS spread is denoted as follows, for every $i = 1, \dots, n$,

$$\kappa_t^i := \kappa_t^{i-1,i} = \frac{\widetilde{P}_t^{i-1,i}}{\widetilde{A}_t^{i-1,i}}, \quad \forall\, t \in [0, T_{i-1}],$$

whereas for the two-period CDS spread we write, for every $i = 2, \dots, n$,

$$\widehat{\kappa}_t^i := \kappa_t^{i-2,i} = \frac{\widetilde{P}_t^{i-2,i}}{\widetilde{A}_t^{i-2,i}}, \quad \forall\, t \in [0, T_{i-2}].$$

Our aim is to derive the semimartingale decompositions for forward CDS spreads $\kappa^1, \dots, \kappa^n$ and $\widehat{\kappa}^2, \dots, \widehat{\kappa}^n$ under a common probability measure. To achieve this, we will first derive expressions for the ratios of defaultable annuities (annuity deflated swap numéraires) in terms of one- and two-period forward CDS spreads.

Lemma 8.5.8. *The following equalities hold, for every $i = 2, \dots, n$ and every $t \in [0, T_{i-2}]$,*

$$\frac{\widetilde{A}_t^{i-2,i-1}}{\widetilde{A}_t^{i-1,i}} = \frac{\widehat{\kappa}_t^i - \kappa_t^i}{\kappa_t^{i-1} - \widehat{\kappa}_t^i}, \tag{8.76}$$

$$\frac{\widetilde{A}_t^{i-2,i}}{\widetilde{A}_t^{i-1,i}} = \frac{\kappa_t^{i-1} - \kappa_t^i}{\kappa_t^{i-1} - \widehat{\kappa}_t^i}. \tag{8.77}$$

Proof. It is easy to check that the following equalities are valid

$$\frac{\widetilde{A}_t^{i-2,i-1}}{\widetilde{A}_t^{i-1,i}} + 1 = \frac{\widetilde{A}_t^{i-2,i}}{\widetilde{A}_t^{i-1,i}},$$

$$\kappa^{i-1} \frac{\widetilde{A}_t^{i-2,i-1}}{\widetilde{A}_t^{i-1,i}} + \kappa^i = \widehat{\kappa}^i \frac{\widetilde{A}_t^{i-2,i}}{\widetilde{A}_t^{i-1,i}},$$

and this in turn yields (8.76) and (8.77). □

In view of formula (8.9) and the assumed positivity of the survival process G, the process $\widehat{A}^{n-1,n}$, which is defined by the formula, for $t \in [0, T_n]$,

$$\widehat{A}_t^{n-1,n} := G_t B_t^{-1} \widetilde{A}_t^{n-1,n} = a_n\, \mathbb{E}_{\mathbb{Q}}(B_{T_n}^{-1} \mathbb{1}_{\{\tau > T_n\}} \,|\, \mathcal{F}_t),$$

is a positive $(\mathbb{Q}, \mathbb{F})$-martingale. Therefore, this process can be used to define the probability measure $\mathbb{P}^n$ on $(\Omega, \mathcal{F}_{T_n})$. Specifically, the probability measure $\mathbb{P}^n$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{T_n})$ is defined by postulating that the Radon-Nikodým density is given by

$$\frac{d\mathbb{P}^n}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{\widehat{A}_t^{n-1,n}}{\widehat{A}_0^{n-1,n}} :\simeq \widehat{A}_t^{n-1,n}, \tag{8.78}$$

where, as before, the symbol $\simeq$ denotes the equality up to a normalizing constant.

We now observe that formulae (8.76) and (8.77) furnish a family of positive processes, which can be used to define a family of equivalent probability measures $\mathbb{P}^i$ for $i = 1, \dots, n$ and $\widehat{\mathbb{P}}^i$ for $i = 2, \dots, n$ such that κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale and $\widehat{\kappa}^i$ is a $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale.

The following commutative diagram summarizes the relationships between these probability measures

$$\begin{array}{ccccccccccc}
\mathbb{Q} & \xrightarrow{\frac{d\mathbb{P}^n}{d\mathbb{Q}}} & \mathbb{P}^n & \xrightarrow{\frac{d\mathbb{P}^{n-1}}{d\mathbb{P}^n}} & \mathbb{P}^{n-1} & \xrightarrow{\frac{d\mathbb{P}^{n-2}}{d\mathbb{P}^{n-1}}} & \dots & \xrightarrow{\frac{d\mathbb{P}^{2}}{d\mathbb{P}^{3}}} & \mathbb{P}^2 & \xrightarrow{\frac{d\mathbb{P}^{1}}{d\mathbb{P}^{2}}} & \mathbb{P}^1 \\
 & & \Big\downarrow{\scriptstyle \frac{d\widehat{\mathbb{P}}^n}{d\mathbb{P}^n}} & & \Big\downarrow{\scriptstyle \frac{d\widehat{\mathbb{P}}^{n-1}}{d\mathbb{P}^{n-1}}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \frac{d\widehat{\mathbb{P}}^2}{d\mathbb{P}^2}} & & \\
 & & \widehat{\mathbb{P}}^n & \xrightarrow{\frac{d\widehat{\mathbb{P}}^{n-1}}{d\widehat{\mathbb{P}}^n}} & \widehat{\mathbb{P}}^{n-1} & \xrightarrow{\frac{d\widehat{\mathbb{P}}^{n-2}}{d\widehat{\mathbb{P}}^{n-1}}} & \dots & \xrightarrow{\frac{d\widehat{\mathbb{P}}^{2}}{d\widehat{\mathbb{P}}^{3}}} & \widehat{\mathbb{P}}^2 & &
\end{array}$$

where we set

$$\begin{aligned}
\frac{d\mathbb{P}^n}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} &:\simeq \widehat{A}_t^{n-1,n}, \\
\frac{d\mathbb{P}^i}{d\mathbb{P}^{i+1}}\Big|_{\mathcal{F}_t} &:\simeq \frac{\widetilde{A}_t^{i-1,i}}{\widetilde{A}_t^{i,i+1}} = \frac{\widehat{\kappa}_t^{i+1} - \kappa_t^{i+1}}{\kappa_t^i - \widehat{\kappa}_t^{i+1}}, \\
\frac{d\widehat{\mathbb{P}}^i}{d\mathbb{P}^i}\Big|_{\mathcal{F}_t} &:\simeq \frac{\widetilde{A}_t^{i-2,i}}{\widetilde{A}_t^{i-1,i}} = \frac{\kappa_t^{i-1} - \kappa_t^i}{\kappa_t^{i-1} - \widehat{\kappa}_t^i}.
\end{aligned}$$

Definition 8.5.3. The probability measure $\mathbb{P}^i$ ($\widehat{\mathbb{P}}^i$, respectively) is called the *martingale measure* for the one-period forward CDS spread κ^i (for the two-period forward CDS spread $\widehat{\kappa}^i$, respectively).

The ultimate goal in the bottom-up approach is the computation of the semimartingale decompositions of processes $\kappa^1, \dots, \kappa^n$ and $\widehat{\kappa}^2, \dots, \widehat{\kappa}^n$ under the martingale measure $\mathbb{P}^n$. In fact, we are not restricted in the choice of the reference probability measure, so any martingale measure can be chosen to play this role. To find the desired decompositions, we observe that it is easy to compute the following Radon-Nikodým densities of martingale measures $\mathbb{P}^i$ and $\widehat{\mathbb{P}}^i$ with respect to the reference probability $\mathbb{P}^n$

$$\begin{aligned}
\frac{d\mathbb{P}^i}{d\mathbb{P}^n}\Big|_{\mathcal{F}_t} &\simeq \frac{\widetilde{A}_t^{i-1,i}}{\widetilde{A}_t^{n-1,n}} = \prod_{j=i+1}^{n} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widehat{\kappa}_t^j}, \\
\frac{d\widehat{\mathbb{P}}^i}{d\mathbb{P}^n}\Big|_{\mathcal{F}_t} &\simeq \frac{\widetilde{A}_t^{i-2,i}}{\widetilde{A}_t^{n-1,n}} = \frac{\widetilde{A}_t^{i-2,i}}{\widetilde{A}_t^{i-1,i}} \frac{\widetilde{A}_t^{i-1,i}}{\widetilde{A}_t^{n-1,n}} = \frac{\kappa_t^{i-1} - \kappa_t^i}{\kappa_t^{i-1} - \widehat{\kappa}_t^i} \prod_{j=i+1}^{n} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widehat{\kappa}_t^j}.
\end{aligned}$$

Explicit formulae for the joint dynamics under $\mathbb{P}^n$ of one- and two-period forward CDS spreads $\kappa^1, \dots, \kappa^n$ and $\widehat{\kappa}^2, \dots, \widehat{\kappa}^n$ are now readily available. Since these dynamics are the same as in the top-down approach presented in the foregoing subsection, we refer to Proposition 8.5.5 below.

Top-Down Approach

In the top-down approach to the modeling of one- and two-period forward CDS spreads, we will work under the following set of standing assumptions. Note that parts (ii), (iii) and (iv) in Assumptions 8.5.5 should be seen as counterparts of the relationships between the martingale measures obtained in the previous subsection within the bottom-up approach.

Assumption 8.5.5. We are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and we postulate that:
(i) the initial values of processes $\kappa^1, \dots, \kappa^n$ and $\widehat{\kappa}^2, \dots, \widehat{\kappa}^n$ are given,
(ii) for every $i = 1, \dots, n$, the process $Z^{\kappa,i}$, which is given by the formula (by the usual convention, $Z^{\kappa,n} = 1$)

$$Z_t^{\kappa,i} = c_i \prod_{j=i+1}^{n} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widehat{\kappa}_t^j}, \tag{8.79}$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale, where c_i is a constant such that $Z_0^{\kappa,i}=1$,
(iii) for every $i=2,\dots,n$, the process $Z^{\widehat{\kappa},i}$, which is given by the formula

$$Z^{\widehat{\kappa},i}=\widehat{c}_i\big(Z^{\kappa,i}+Z^{\kappa,i-1}\big)=\widehat{c}_i\frac{\kappa^{i-1}-\kappa^i}{\kappa^{i-1}-\widehat{\kappa}^i}\,Z^{\kappa,i}, \tag{8.80}$$

is a positive $(\mathbb{P},\mathbb{F})$-martingale, where $\widehat{c}_i$ is a constant such that $Z_0^{\widehat{\kappa},i}=1$,
(iv) for every $i=1,\dots,n$, the process κ^i is a $(\mathbb{P}^i,\mathbb{F})$-martingale, where the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{\kappa,i}$,
(v) for every $i=1,\dots,n$, the process κ^i satisfies

$$\kappa_t^i=\kappa_0^i+\int_{(0,t]}\kappa_{s-}^i\widetilde{\sigma}_s^i\cdot d\widetilde{N}_s^i, \tag{8.81}$$

where $\widetilde{N}^i$ is an $\mathbb{R}^{k_i}$-valued $(\mathbb{P}^i,\mathbb{F})$-martingale and $\widetilde{\sigma}^i$ is an $\mathbb{R}^{k_i}$-valued, $\mathbb{F}$-predictable, $\widetilde{N}^i$-integrable process,
(vi) for every $i=2,\dots,n$, the process $\widehat{\kappa}^i$ is a $(\widehat{\mathbb{P}}^i,\mathbb{F})$-martingale, where the Radon-Nikodým density of $\widehat{\mathbb{P}}^i$ with respect to $\mathbb{P}$ equals $Z^{\widehat{\kappa},i}$,
(vii) for every $i=2,\dots,n$, the process $\widehat{\kappa}^i$ satisfies

$$\widehat{\kappa}_t^i=\widehat{\kappa}_0^i+\int_{(0,t]}\widehat{\kappa}_{s-}^i\widehat{\zeta}_s^i\cdot d\widehat{N}_s^i, \tag{8.82}$$

where $\widehat{N}^i$ is an $\mathbb{R}^{l_i}$-valued $(\widehat{\mathbb{P}}^i,\mathbb{F})$-martingale and $\widehat{\zeta}^i$ is an $\mathbb{R}^{l_i}$-valued, $\mathbb{F}$-predictable, $\widehat{N}^i$-integrable process.

The probability measure $\mathbb{P}^i$ ($\widehat{\mathbb{P}}^i$, respectively) is the martingale measure for the one-period forward CDS spread κ^i (for the two-period forward CDS spread $\widehat{\kappa}^i$, respectively). It is worth noting that the equality $\mathbb{P}^n=\mathbb{P}$ holds, since $Z^{\kappa,n}=1$, so that the underlying probability measure $\mathbb{P}$ is the martingale measure for the process κ^n.

Let $k=k_1+\dots+k_n+l_2+\dots+l_n$. As in preceding subsections, we argue that there exists an $\mathbb{R}^k$-valued $(\mathbb{P},\mathbb{F})$-martingale M such that for every $i=1,\dots,n$ the process κ^i admits the following representation under $\mathbb{P}^i$

$$\kappa_t^i=\kappa_0^i+\int_{(0,t]}\kappa_{s-}^i\sigma_s^i\cdot d\Psi^i(M)_s, \tag{8.83}$$

and for every $i=2,\dots,n$ the process $\widehat{\kappa}^i$ admits the following representation under $\widehat{\mathbb{P}}^i$

$$\widehat{\kappa}_t^i=\widehat{\kappa}_0^i+\int_{(0,t]}\widehat{\kappa}_{s-}^i\zeta_s^i\cdot d\widehat{\Psi}^i(M)_s, \tag{8.84}$$

where $\sigma^i=(\sigma^{i,1},\dots,\sigma^{i,k})$ and $\zeta^i=(\zeta^{i,1},\dots,\zeta^{i,k})$ are $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable processes, which extend $\widetilde{\sigma}^i$ and $\widehat{\zeta}^i$, respectively. The $(\mathbb{P}^i,\mathbb{F})$-martingale $\Psi^i(M^l)$ is given by

$$\Psi^i(M^l)_t=M_t^l-[\mathcal{L}(Z^{\kappa,i})^c,M^{l,c}]_t-\sum_{0<s\le t}\frac{1}{Z_s^{\kappa,i}}\Delta Z_s^{\kappa,i}\Delta M_s^l,$$

whereas the $(\widehat{\mathbb{P}}^i,\mathbb{F})$-martingale $\widehat{\Psi}^i(M^l)$ equals

$$\widehat{\Psi}^i(M^l)_t=M_t^l-\big[\mathcal{L}(Z^{\widehat{\kappa},i})^c,M^{l,c}\big]_t-\sum_{0<s\le t}\frac{1}{Z_s^{\widehat{\kappa},i}}\Delta Z_s^{\widehat{\kappa},i}\Delta M_s^l.$$

We note that since $Z^{\kappa,n}=1$, we have that $\Psi^n(M)=M$ and thus (8.83) yields

$$\kappa_t^n=\kappa_0^n+\int_{(0,t]}\kappa_{s-}^n\sigma_s^n\cdot dM_s.$$

Let us observe that the present situation corresponds to Setup (C) introduced in Subsection 8.4.3. The following result is a rather straightforward consequence of Proposition 8.4.3 and thus its proof is omitted.

Lemma 8.5.9. *Under Assumptions 8.5.5, for every $i = 1, \ldots, n$ the semimartingale decomposition of the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ under the martingale measure $\mathbb{P} = \mathbb{P}^n$ is given by*

$$\begin{aligned}\Psi^i(M)_t = M_t &- \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\left(\widehat{\kappa}^j_s \zeta^j_s - \kappa^j_s \sigma^j_s\right) \cdot d[M^c]_s}{\widehat{\kappa}^j_s - \kappa^j_s} + \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\left(\kappa^{j-1}_s \sigma^{j-1}_s - \widehat{\kappa}^j_s \zeta^j_s\right) \cdot d[M^c]_s}{\kappa^{j-1}_s - \widehat{\kappa}^j_s} \\ &- \sum_{0<s\le t} \frac{1}{Z^{\kappa,i}_s} \Delta Z^{\kappa,i}_s \Delta M_s,\end{aligned}$$

and for every $i = 2, \ldots, n$ the semimartingale decomposition of the $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale $\widehat{\Psi}^i(M)$ under the martingale measure $\mathbb{P} = \mathbb{P}^n$ is given by

$$\begin{aligned}\widehat{\Psi}^i(M)_t = M_t &- \int_{(0,t]} \frac{\left(\kappa^{i-1}_s \sigma^{i-1}_s - \kappa^i_s \sigma^i_s\right) \cdot d[M^c]_s}{\left(\kappa^{i-1}_s - \kappa^i_s\right)} + \int_{(0,t]} \frac{\left(\kappa^{i-1}_s \sigma^{i-1}_s - \widehat{\kappa}^i_s \zeta^i_s\right) \cdot d[M^c]_s}{\kappa^{i-1}_s - \widehat{\kappa}^i_s} \\ &- \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\left(\widehat{\kappa}^j_s \zeta^j_s - \kappa^j_s \sigma^j_s\right) \cdot d[M^c]_s}{\widehat{\kappa}^j_s - \kappa^j_s} + \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\left(\kappa^{j-1}_s \sigma^{j-1}_s - \widehat{\kappa}^j_s \zeta^j_s\right) \cdot d[M^c]_s}{\kappa^{j-1}_s - \widehat{\kappa}^j_s} \\ &- \sum_{0<s\le t} \frac{1}{Z^{\widehat{\kappa},i}_s} \Delta Z^{\widehat{\kappa},i}_s \Delta M_s.\end{aligned}$$

To obtain an explicit expression for the joint dynamics of one-period spreads $\kappa^1, \ldots, \kappa^n$ and two-period spreads $\widehat{\kappa}^2, \ldots, \widehat{\kappa}^n$ under the probability measure $\mathbb{P} = \mathbb{P}^n$, it suffices to combine Lemma 8.5.9 with formulae (8.83) and (8.84). Note that the system of stochastic differential equations given in Proposition 8.5.5 is closed, in the sense that as soon as the driving $(\mathbb{P}, \mathbb{F})$-martingale M and the volatility processes $\sigma^1, \ldots, \sigma^n$ and $\zeta^2, \ldots, \zeta^n$ are chosen, the joint dynamics of the processes $\kappa^1, \ldots, \kappa^n, \widehat{\kappa}^2, \ldots, \widehat{\kappa}^n$ under the martingale measure $\mathbb{P}^n$ are uniquely specified. After some standard computations, one can also deduce from this result the joint dynamics of these processes under any martingale measure $\mathbb{P}^i$ or $\widehat{\mathbb{P}}^i$.

Proposition 8.5.5. *Under Assumptions 8.5.5, the joint dynamics of one-period forward CDS spreads $\kappa^1, \ldots, \kappa^n$ and two-period forward CDS spreads $\widehat{\kappa}^2, \ldots, \widehat{\kappa}^n$ under the martingale measure $\mathbb{P} = \mathbb{P}^n$ are given by the following $(2n-1)$-dimensional stochastic differential equation driven by the $(\mathbb{P}, \mathbb{F})$-martingale M: for every $i = 1, \ldots, n$ and $t \in [0, T_{i-1}]$*

$$\begin{aligned}d\kappa^i_t = \sum_{l=1}^{k} \kappa^i_{t-} \sigma^{i,l}_t \, dM^l_t &- \sum_{j=i+1}^{n} \frac{\kappa^i_t \widehat{\kappa}^j_t}{\widehat{\kappa}^j_t - \kappa^j_t} \sum_{l,m=1}^{k} \sigma^{i,l}_t \zeta^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\ &+ \sum_{j=i+1}^{n} \frac{\kappa^i_t \kappa^j_t}{\widehat{\kappa}^j_t - \kappa^j_t} \sum_{l,m=1}^{k} \sigma^{i,l}_t \sigma^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\ &+ \sum_{j=i+1}^{n} \frac{\kappa^i_t \kappa^{j-1}_t}{\kappa^{j-1}_t - \widehat{\kappa}^j_t} \sum_{l,m=1}^{k} \sigma^{i,l}_t \sigma^{j-1,m}_t \, d[M^{l,c}, M^{m,c}]_t \\ &- \sum_{j=i+1}^{n} \frac{\kappa^i_t \widehat{\kappa}^j_t}{\kappa^{j-1}_t - \widehat{\kappa}^j_t} \sum_{l,m=1}^{k} \sigma^{i,l}_t \zeta^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\ &- \frac{\kappa^i_{t-}}{Z^{\kappa,i}_t} \Delta Z^{\kappa,i}_t \sum_{l=1}^{k} \sigma^{i,l}_t \Delta M^l_t\end{aligned}$$

and for every $i = 2, \ldots, n$ and $t \in [0, T_{i-2}]$

$$\begin{aligned}
d\widehat{\kappa}^i_t &= \sum_{l=1}^k \widehat{\kappa}^i_{t-} \zeta^{i,l}_t \, dM^l_t - \frac{\widehat{\kappa}^i_t \kappa^{i-1}_t}{\kappa^{i-1}_t - \kappa^i_t} \sum_{l,m=1}^k \zeta^{i,l}_t \sigma^{i-1,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&\quad + \frac{\widehat{\kappa}^i_t \kappa^i_t}{\kappa^{i-1}_t - \kappa^i_t} \sum_{l,m=1}^k \zeta^{i,l}_t \sigma^{i,m}_t \, d[M^{l,c}, M^{m,c}]_t + \frac{\widehat{\kappa}^i_t \kappa^{i-1}_t}{\kappa^{i-1}_t - \widehat{\kappa}^i_t} \sum_{l,m=1}^k \zeta^{i,l}_t \sigma^{i-1,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&\quad - \frac{\widehat{\kappa}^i_t \widehat{\kappa}^i_t}{\kappa^{i-1}_t - \widehat{\kappa}^i_t} \sum_{l,m=1}^k \zeta^{i,l}_t \zeta^{i,m}_t \, d[M^{l,c}, M^{m,c}]_t - \sum_{j=i+1}^n \frac{\widehat{\kappa}^i_t \widehat{\kappa}^j_t}{\widehat{\kappa}^j_t - \kappa^j_t} \sum_{l,m=1}^k \zeta^{i,l}_t \zeta^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&\quad + \sum_{j=i+1}^n \frac{\widehat{\kappa}^i_t \kappa^j_t}{\widehat{\kappa}^j_t - \kappa^j_t} \sum_{l,m=1}^k \zeta^{i,l}_t \sigma^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t + \sum_{j=i+1}^n \frac{\widehat{\kappa}^i_t \kappa^{j-1}_t}{\kappa^{j-1}_t - \widehat{\kappa}^j_t} \sum_{l,m=1}^k \zeta^{i,l}_t \sigma^{j-1,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
&\quad - \sum_{j=i+1}^n \frac{\widehat{\kappa}^i_t \widehat{\kappa}^j_t}{\kappa^{j-1}_t - \widehat{\kappa}^j_t} \sum_{l,m=1}^k \zeta^{i,l}_t \zeta^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t - \frac{\widehat{\kappa}^i_{t-}}{Z^{\widehat{\kappa},i}_t} \, \Delta Z^{\widehat{\kappa},i}_t \sum_{l=1}^k \zeta^{i,l}_t \Delta M^l_t.
\end{aligned}$$

Unfortunately, we were not able to show the uniqueness of a solution to the system of SDEs derived in Proposition 8.5.5, since the coefficients are not locally Lipschitz continuous (a similar comment applies to the system of SDEs obtained in Proposition 8.5.6 below).

8.5.4 Model of One-Period and Co-Terminal CDS Spreads

In this subsection, we will study the market model for forward CDS spreads associated with a family of one-period and co-terminal CDSs. Such a model can be seen as a counterpart of the model of co-terminal forward swap rates, which was proposed by Jamshidian [53] as an alternative for the model of forward LIBORs as a tool for valuation of exotic swap derivatives, such as Bermudan swaptions. In the context of CDS spreads, this model was suggested in the paper by Brigo [23]. In this setup, the goal is to explicitly determine the dynamics of the family of one-period forward CDS spreads and the co-terminal forward CDS spreads $\kappa^{0,n}, \kappa^{1,n}, \ldots, \kappa^{n-2,n}$ under a common probability measure. As model inputs, we take the driving martingales and a collection of volatility processes for the considered family of forward CDS spreads. The importance of this particular family of forward CDS spreads becomes clear when dealing with the valuation of Bermudan CDS options.

Bottom-Up Approach

It should be emphasized that throughout this section all CDS spreads are indexed by the starting date, rather than the maturity date (unlike in Section 8.5.3 where a different convention was adopted). The one-period forward CDS spread is thus denoted as

$$\kappa^i_t := \kappa^{i,i+1}_t = \frac{\widetilde{P}^{i,i+1}_t}{\widetilde{A}^{i,i+1}_t}, \quad \forall \, t \in [0, T_i],$$

for every $i = 0, \ldots, n-1$. The co-terminal CDS spread with start date T_i and maturity T_n will be denoted by

$$\widehat{\kappa}^i_t := \kappa^{i,n}_t = \frac{\widetilde{P}^{i,n}_t}{\widetilde{A}^{i,n}_t}, \quad \forall \, t \in [0, T_i],$$

for $i = 0, \ldots, n-1$. It is thus clear that the equality $\kappa^{n-1} = \widehat{\kappa}^{n-1}$ holds. The goal of this notation is to simplify the presentation and emphasize some formal similarities between the present model and the model examined in the preceding subsection.

We wish to derive the semimartingale decompositions for processes $\kappa^0, \ldots, \kappa^{n-2}$ and $\widehat{\kappa}^0, \ldots, \widehat{\kappa}^{n-1}$ under a single probability measure. As in preceding subsections, we need first to express the ratios of defaultable annuities (annuity deflated swap numéraires) in terms of the underlying family of forward CDS spreads.

Lemma 8.5.10. *The following equalities hold, for every $i = 0, \dots, n-2$ and every $t \in [0, T_i]$,*

$$\frac{\widetilde{A}_t^{i+1,n}}{\widetilde{A}_t^{i,n}} = \frac{\widehat{\kappa}_t^i - \kappa_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i}, \tag{8.85}$$

$$\frac{\widetilde{A}_t^{i,i+1}}{\widetilde{A}_t^{i,n}} = \frac{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i}. \tag{8.86}$$

Proof. It is easy to check that the following relationships hold true

$$\frac{\widetilde{A}^{i,i+1}}{\widetilde{A}^{i,n}} + \frac{\widetilde{A}^{i+1,n}}{\widetilde{A}^{i,n}} = 1,$$

$$\kappa^i \frac{\widetilde{A}^{i,i+1}}{\widetilde{A}^{i,n}} + \widehat{\kappa}^{i+1} \frac{\widetilde{A}^{i+1,n}}{\widetilde{A}^{i,n}} = \widehat{\kappa}^i.$$

By solving this system, we obtain (8.85) and (8.86). □

It is easily seen from (8.9) that the process $\widehat{A}^{0,n}$, which is given by the formula, for every $t \in [0, T_0]$,

$$\widehat{A}_t^{0,n} := G_t B_t^{-1} \widetilde{A}_t^{0,n} = \sum_{j=1}^{n} a_j \, \mathbb{E}_{\mathbb{Q}}(B_{T_j}^{-1} \mathbb{1}_{\{\tau > T_j\}} \,|\, \mathcal{F}_t),$$

is a positive $(\mathbb{Q}, \mathbb{F})$-martingale. Therefore, it can be used to define the probability measure $\widehat{\mathbb{P}}^0$ on $(\Omega, \mathcal{F}_{T_0})$. Moreover, formulae (8.85) and (8.86) furnish a family of positive processes, which can be used to define a family of probability measures $\mathbb{P}^i$ for $i = 0, \dots, n-2$ and $\widehat{\mathbb{P}}^i$ for $i = 0, \dots, n-1$, such that κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale and $\widehat{\kappa}^i$ is a $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale for $t \in [0, T_0]$.

The relationships between these martingale measures are summarized in the following commutative diagram

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, "\frac{d\widehat{\mathbb{P}}^0}{d\mathbb{Q}}"] & \widehat{\mathbb{P}}^0 \arrow[r, "\frac{d\widehat{\mathbb{P}}^1}{d\widehat{\mathbb{P}}^0}"] \arrow[d, "\frac{d\mathbb{P}^0}{d\widehat{\mathbb{P}}^0}"'] & \widehat{\mathbb{P}}^1 \arrow[r, "\frac{d\widehat{\mathbb{P}}^2}{d\widehat{\mathbb{P}}^1}"] \arrow[d, "\frac{d\mathbb{P}^1}{d\widehat{\mathbb{P}}^1}"'] & \dots \arrow[r, "\frac{d\widehat{\mathbb{P}}^{n-2}}{d\widehat{\mathbb{P}}^{n-3}}"] \arrow[d] & \widehat{\mathbb{P}}^{n-2} \arrow[r, "\frac{d\widehat{\mathbb{P}}^{n-1}}{d\widehat{\mathbb{P}}^{n-2}}"] \arrow[d, "\frac{d\mathbb{P}^{n-2}}{d\widehat{\mathbb{P}}^{n-2}}"'] & \widehat{\mathbb{P}}^{n-1} = \mathbb{P}^{n-1} \\
 & \mathbb{P}^0 \arrow[r, "\frac{d\mathbb{P}^1}{d\mathbb{P}^0}"] & \mathbb{P}^1 \arrow[r, "\frac{d\mathbb{P}^2}{d\mathbb{P}^1}"] & \dots \arrow[r, "\frac{d\mathbb{P}^{n-2}}{d\mathbb{P}^{n-3}}"] & \mathbb{P}^{n-2} \arrow[r, "\frac{d\mathbb{P}^{n-1}}{d\mathbb{P}^{n-2}}"] & \mathbb{P}^{n-1}
\end{tikzcd}$$

where we set

$$\frac{d\widehat{\mathbb{P}}^0}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} :\simeq \widehat{A}_t^{0,n},$$

$$\frac{d\widehat{\mathbb{P}}^{i+1}}{d\widehat{\mathbb{P}}^i}\Big|_{\mathcal{F}_t} :\simeq \frac{\widetilde{A}_t^{i+1,n}}{\widetilde{A}_t^{i,n}} = \frac{\widehat{\kappa}_t^i - \kappa_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i},$$

$$\frac{d\mathbb{P}^i}{d\widehat{\mathbb{P}}^i}\Big|_{\mathcal{F}_t} :\simeq \frac{\widetilde{A}_t^{i,i+1}}{\widetilde{A}_t^{i,n}} = \frac{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i},$$

where, as usual, $\simeq$ denotes equality up to a normalizing constant. Using these equalities, we also obtain the following family of the Radon-Nikodým densities, for every $i = 0, \dots, n-1$ and every $t \in [0, T_0]$,

$$\frac{d\widehat{\mathbb{P}}^i}{d\widehat{\mathbb{P}}^0}\Big|_{\mathcal{F}_t} :\simeq \frac{\widetilde{A}_t^{i,n}}{\widetilde{A}_t^{0,n}} = \prod_{j=0}^{i-1} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j},$$

and for every $i = 0, \dots, n-2$ and every $t \in [0, T_0]$,

$$\frac{d\mathbb{P}^i}{d\widehat{\mathbb{P}}^0}\Big|_{\mathcal{F}_t} :\simeq \frac{\widetilde{A}_t^{i,i+1}}{\widetilde{A}_t^{0,n}} = \frac{\widetilde{A}_t^{i,i+1}}{\widetilde{A}_t^{i,n}} \frac{\widetilde{A}_t^{i,n}}{\widetilde{A}_t^{0,n}} = \frac{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i} \prod_{j=0}^{i-1} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j}.$$

It is now not difficult to derive the joint dynamics under $\widehat{\mathbb{P}}^0$ of processes $\kappa^0, \dots, \kappa^{n-2}$ and $\widehat{\kappa}^0, \dots, \widehat{\kappa}^{n-1}$. However, since they are the same as in the top-down approach presented in the next subsection, we refer the reader to Proposition 8.5.6 below.

Top-Down Approach

In this subsection, we present the top-down approach to the modeling of one-period and co-terminal forward CDS spreads. As in the preceding subsections, we start by stating a number of postulates underpinning the top-down approach. They are motivated by the bottom-down approach, meaning that they should be seen as necessary conditions for the consistency of the top-down model. The goal of the top-down approach is to show that Assumptions 8.5.6 are also sufficient for the unique specification of the joint dynamics for the family of forward CDS spreads considered in this subsection.

Assumption 8.5.6. We are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and we postulate that:
(i) the initial values of processes $\kappa^0, \dots, \kappa^{n-2}$ and $\widehat{\kappa}^0, \dots, \widehat{\kappa}^{n-1}$ are given,
(ii) for every $i = 0, \dots, n-1$, the process $Z^{\widehat{\kappa},i}$, which is given by the formula (by the usual convention, $Z^{\widehat{\kappa},0} = 1$)

$$Z_t^{\widehat{\kappa},i} = \widehat{c}_i \prod_{j=0}^{i-1} \frac{\widehat{\kappa}_t^j - \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j}, \quad \forall\, t \in [0, T_0],$$

is a positive $(\mathbb{P}, \mathbb{F})$-martingale and $\widehat{c}_i$ is a constant such that $Z_0^{\widehat{\kappa},i} = 1$,
(iii) for every $i = 0, \dots, n-2$, the process $Z^{\kappa,i}$, which is given by the formula

$$Z_t^{\kappa,i} = c_i \frac{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i}\, Z_t^{\widehat{\kappa},i}, \quad \forall\, t \in [0, T_0],$$

is a positive $(\mathbb{P}, \mathbb{F})$-martingale and c_i is a constant such that $Z_0^{\kappa,i} = 1$,
(iv) for every $i = 0, \dots, n-2$, the process κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale, where the Radon-Nikodým density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ equals $Z^{\kappa,i}$,
(v) for every $i = 0, \dots, n-2$, the process κ^i satisfies

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \kappa_{s-}^i \widetilde{\sigma}_s^i \cdot d\widetilde{N}_s^i, \tag{8.87}$$

where $\widetilde{N}^i$ is an $\mathbb{R}^{k_i}$-valued $(\mathbb{P}^i, \mathbb{F})$-martingale and $\widetilde{\sigma}^i$ is an $\mathbb{R}^{k_i}$-valued, $\mathbb{F}$-predictable, $\widetilde{N}^i$-integrable process,
(vi) for every $i = 0, \dots, n-1$, the process $\widehat{\kappa}^i$ is a $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale, where the Radon-Nikodým density of $\widehat{\mathbb{P}}^i$ with respect to $\mathbb{P}$ equals $Z^{\widehat{\kappa},i}$,
(vii) for every $i = 0, \dots, n-1$, the process $\widehat{\kappa}^i$ satisfies

$$\widehat{\kappa}_t^i = \widehat{\kappa}_0^i + \int_{(0,t]} \widehat{\kappa}_{s-}^i \widehat{\zeta}_s^i \cdot d\widehat{N}_s^i, \tag{8.88}$$

where $\widehat{N}^i$ is an $\mathbb{R}^{l_i}$-valued $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale and $\widehat{\zeta}^i$ is an $\mathbb{R}^{l_i}$-valued, $\mathbb{F}$-predictable, $\widehat{N}^i$-integrable process.

As usual, we argue that the volatilities $\widetilde{\sigma}^i$ and $\widehat{\zeta}^i$ can be chosen arbitrarily, provided, of course, that the stochastic differential equations (8.87) and (8.88) have unique solutions. Note also that the equality $\widehat{\mathbb{P}}^0 = \mathbb{P}$ holds, since $Z^{\widehat{\kappa},0} = 1$.

Let $k = k_0 + \cdots + k_{n-2} + l_0 + \cdots + l_{n-1}$. It is not difficult to show that there exists an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale M such that for every $i = 0, \ldots, n-2$, the $(\mathbb{P}^i, \mathbb{F})$-martingale κ^i admits the following representation under $\mathbb{P}^i$

$$\kappa^i_t = \kappa^i_0 + \int_{(0,t]} \kappa^i_{s-} \sigma^i_s \cdot d\Psi^i(M)_s, \tag{8.89}$$

where $\sigma^i = (\sigma^{i,1}, \ldots, \sigma^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process, which extends $\widetilde{\sigma}^i$. Similarly, the $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale $\widehat{\kappa}^i$ has the following representation under $\widehat{\mathbb{P}}^i$, for every $i = 0, \ldots, n-1$,

$$\widehat{\kappa}^i_t = \widehat{\kappa}^i_0 + \int_{(0,t]} \widehat{\kappa}^i_{s-} \zeta^i_s \cdot d\widehat{\Psi}^i(M)_s, \tag{8.90}$$

where $\zeta^i = (\zeta^{i,1}, \ldots, \zeta^{i,k})$ is an $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process, which extends $\widehat{\zeta}^i$. Furthermore, the $\mathbb{R}^k$-valued $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ satisfies, for every $l = 1, \ldots, k$,

$$\Psi^i(M^l)_t = M^l_t - \big[\mathcal{L}(Z^{\kappa,i})^c, M^{l,c}\big]_t - \sum_{0<s\le t} \frac{1}{Z^{\kappa,i}_s} \Delta Z^{\kappa,i}_s \Delta M^l_s,$$

whereas the $\mathbb{R}^k$-valued $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale $\widehat{\Psi}^i(M)$ satisfies, for every $l = 1, \ldots, k$,

$$\widehat{\Psi}^i(M^l)_t = M^l_t - \big[\mathcal{L}(Z^{\widehat{\kappa},i})^c, M^{l,c}\big]_t - \sum_{0<s\le t} \frac{1}{Z^{\widehat{\kappa},i}_s} \Delta Z^{\widehat{\kappa},i}_s \Delta M^l_s.$$

In particular, $\mathbb{P} = \widehat{\mathbb{P}}^0$ and thus $\widehat{\Psi}^0(M) = M$. Consequently, the forward CDS spread $\widehat{\kappa}^0$ satisfies

$$\widehat{\kappa}^0_t = \widehat{\kappa}^0_0 + \int_{(0,t]} \widehat{\kappa}^0_{s-} \zeta^0_s \cdot dM_s. \tag{8.91}$$

The equality (8.91) agrees with the fact that $\widehat{\mathbb{P}}^0$ is the martingale measure for $\widehat{\kappa}^0$.

It is not difficult to verify that the current assumptions place us within Setup (D) examined in Subsection 8.4.4. The following result is a rather straightforward consequence of Proposition 8.4.4, so that its proof is omitted.

Lemma 8.5.11. *Under Assumptions 8.5.6, for every $i = 0, \ldots, n-2$ the semimartingale decomposition of the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ under the martingale measure $\mathbb{P} = \widehat{\mathbb{P}}^0$ is given by*

$$\begin{aligned}
\Psi^i(M)_t = M_t &- \int_{(0,t]} \frac{\big(\widehat{\kappa}^{i+1}_s \zeta^{i+1}_s - \widehat{\kappa}^i_s \zeta^i_s\big) \cdot d[M^c]_s}{(\widehat{\kappa}^{i+1}_s - \widehat{\kappa}^i_s)} + \int_{(0,t]} \frac{\big(\widehat{\kappa}^{i+1}_s \zeta^{i+1}_s - \kappa^i_s \sigma^i_s\big) \cdot d[M^c]_s}{\widehat{\kappa}^{i+1}_s - \kappa^i_s} \\
&- \sum_{j=0}^{i-1} \int_{(0,t]} \frac{\big(\widehat{\kappa}^j_s \zeta^j_s - \kappa^j_s \sigma^j_s\big) \cdot d[M^c]_s}{\widehat{\kappa}^j_s - \kappa^j_s} + \sum_{j=0}^{i-1} \int_{(0,t]} \frac{\big(\widehat{\kappa}^{j+1}_s \sigma^{j+1}_s - \kappa^j_s \sigma^j_s\big) \cdot d[M^c]_s}{\widehat{\kappa}^{j+1}_s - \kappa^j_s} \\
&- \sum_{0<s\le t} \frac{1}{Z^{\kappa,i}_s} \Delta Z^{\kappa,i}_s \Delta M_s,
\end{aligned}$$

and for every $i = 0, \ldots, n-1$ the semimartingale decomposition of the $(\widehat{\mathbb{P}}^i, \mathbb{F})$-martingale $\widehat{\Psi}^i(M)$ under the martingale measure $\mathbb{P} = \widehat{\mathbb{P}}^0$ is given by

$$\begin{aligned}
\widehat{\Psi}^i(M)_t = M_t &- \sum_{j=0}^{i-1} \int_{(0,t]} \frac{\big(\widehat{\kappa}^j_s \zeta^j_s - \kappa^j_s \sigma^j_s\big) \cdot d[M^c]_s}{\widehat{\kappa}^j_s - \kappa^j_s} + \sum_{j=0}^{i-1} \int_{(0,t]} \frac{\big(\widehat{\kappa}^{j+1}_s \zeta^{j+1}_s - \kappa^j_s \sigma^j_s\big) \cdot d[M^c]_s}{\widehat{\kappa}^{j+1}_s - \kappa^j_s} \\
&- \sum_{0<s\le t} \frac{1}{Z^{\widehat{\kappa},i}_s} \Delta Z^{\widehat{\kappa},i}_s \Delta M_s.
\end{aligned}$$

By combining Lemma 8.5.11 with formulae (8.89) and (8.90), we obtain closed-form expressions for semimartingale decompositions of the family of one-period forward CDS spreads $\kappa^0, \dots, \kappa^{n-2}$ and the co-terminal forward CDS spreads $\widehat{\kappa}^0, \dots, \widehat{\kappa}^{n-1}$ under the probability measure $\mathbb{P} = \widehat{\mathbb{P}}^0$, that is, under the martingale measure associated with the n-period forward CDS spread $\widehat{\kappa}^0$. We now assume that the driving $(\mathbb{P}, \mathbb{F})$-martingale M and the volatility processes $\sigma^0, \dots, \sigma^{n-2}$ $\zeta^0, \dots, \zeta^{n-1}$ are given. It is also worth recalling that $\kappa^{n-1} = \widehat{\kappa}^{n-1}$.

Proposition 8.5.6. *Under Assumptions 8.5.6, the joint dynamics of one-period forward CDS spreads $\kappa^0, \dots, \kappa^{n-2}$ and co-terminal forward CDS spreads $\widehat{\kappa}^0, \dots, \widehat{\kappa}^{n-1}$ under the martingale measure $\mathbb{P} = \widehat{\mathbb{P}}^0$ are given by the following $(2n-1)$-dimensional stochastic differential equation driven by the $(\mathbb{P}, \mathbb{F})$-martingale M: for every $i = 0, \dots, n-2$ and $t \in [0, T_0]$*

$$
\begin{aligned}
d\kappa_t^i = {} & \sum_{l=1}^{k} \kappa_{t-}^i \sigma_t^{i,l}\, dM_t^l - \frac{\kappa_t^i \widehat{\kappa}_t^{i+1}}{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{i+1,m}\, d[M^{l,c}, M^{m,c}]_t \\
& + \frac{\kappa_t^i \widehat{\kappa}_t^i}{\widehat{\kappa}_t^{i+1} - \widehat{\kappa}_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{i,m}\, d[M^{l,c}, M^{m,c}]_t + \frac{\kappa_t^i \widehat{\kappa}_t^{i+1}}{\widehat{\kappa}_t^{i+1} - \kappa_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{i+1,m}\, d[M^{l,c}, M^{m,c}]_t \\
& - \frac{\kappa_t^i \kappa_t^i}{\widehat{\kappa}_t^{i+1} - \kappa_t^i} \sum_{l,m=1}^{k} \sigma_t^{i,l} \sigma_t^{i,m}\, d[M^{l,c}, M^{m,c}]_t - \sum_{j=0}^{i-1} \frac{\kappa_t^i \widehat{\kappa}_t^j}{\widehat{\kappa}_t^j - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\
& + \sum_{j=0}^{i-1} \frac{\kappa_t^i \kappa_t^j}{\widehat{\kappa}_t^j - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t + \sum_{j=0}^{i-1} \frac{\kappa_t^i \widehat{\kappa}_t^{j+1}}{\widehat{\kappa}_t^{j+1} - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \zeta_t^{j+1,m}\, d[M^{l,c}, M^{m,c}]_t \\
& - \sum_{j=0}^{i-1} \frac{\kappa_t^i \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \kappa_t^j} \sum_{l,m=1}^{k} \sigma_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t - \frac{\kappa_{t-}^i}{Z_t^{\widehat{\kappa},i}} \Delta Z_t^{\widehat{\kappa},i} \sum_{l=1}^{k} \sigma_t^{i,l} \Delta M_t^l
\end{aligned}
$$

and for every $i = 0, \dots, n-1$ and $t \in [0, T_0]$

$$
\begin{aligned}
d\widehat{\kappa}_t^i = {} & \sum_{l=1}^{k} \widehat{\kappa}_{t-}^i \zeta_t^{i,l}\, dM_t^l - \sum_{j=0}^{i-1} \frac{\widehat{\kappa}_t^i \widehat{\kappa}_t^j}{\widehat{\kappa}_t^j - \kappa_t^j} \sum_{l,m=1}^{k} \zeta_t^{i,l} \zeta_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\
& + \sum_{j=0}^{i-1} \frac{\widehat{\kappa}_t^i \kappa_t^j}{\widehat{\kappa}_t^j - \kappa_t^j} \sum_{l,m=1}^{k} \zeta_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\
& + \sum_{j=0}^{i-1} \frac{\widehat{\kappa}_t^i \widehat{\kappa}_t^{j+1}}{\widehat{\kappa}_t^{j+1} - \widehat{\kappa}_t^j} \sum_{l,m=1}^{k} \zeta_t^{i,l} \zeta_t^{j+1,m}\, d[M^{l,c}, M^{m,c}]_t \\
& - \sum_{j=0}^{i-1} \frac{\widehat{\kappa}_t^i \kappa_t^j}{\widehat{\kappa}_t^{j+1} - \widehat{\kappa}_t^j} \sum_{l,m=1}^{k} \zeta_t^{i,l} \sigma_t^{j,m}\, d[M^{l,c}, M^{m,c}]_t \\
& - \frac{\widehat{\kappa}_{t-}^i}{Z_t^{\kappa,i}} \Delta Z_t^{\kappa,i} \sum_{l=1}^{k} \zeta_t^{i,l} \Delta M_t^l.
\end{aligned}
$$

Alternatively, one can derive the joint dynamics of forward CDS spreads under the martingale measure $\mathbb{P}^n = \widehat{\mathbb{P}}^n$ (see Rutkowski [100]) or, whenever this is desirable, under any other martingale measure $\mathbb{P}^i$ or $\widehat{\mathbb{P}}^i$ from the family introduced in this subsection. The most convenient choice of the underlying probability measure depends on a particular problem at hand. Our choice of the reference probability measure $\mathbb{P} = \widehat{\mathbb{P}}^0$ was motivated by desire to emphasize the symmetrical features of setups considered in Subsections 8.5.3 and 8.5.4, rather than by practical considerations.

8.5.5 Generic Top-Down Approach

We will now make an attempt to describe the main steps in the top-down approach to a generic family of forward spreads. We now consider a generic family of forward swap rates and/or forward CDS spreads, denoted as $\kappa^1, \ldots, \kappa^n$, and we are interested in the top-down approach to the derivation of the joint dynamics of these processes. The generic top-down method hinges on the following set of postulates.

Assumption 8.5.7. We are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$. We postulate that the $\mathbb{F}$-adapted processes $\kappa^1, \ldots, \kappa^n$ and $Z^1, \ldots, Z^n$ satisfy the following conditions, for every $i = 1, \ldots, n$,
(i) the process Z^i is a positive $(\mathbb{P}, \mathbb{F})$-martingale with $\mathbb{E}_{\mathbb{P}}(Z_0^i) = 1$,
(ii) the process $\kappa^i Z^i$ is a $(\mathbb{P}, \mathbb{F})$-martingale,
(iii) the process Z^i is given as a function of some subset of the family $\kappa^1, \ldots, \kappa^n$; specifically, there exists a subset $\{\kappa^{n_1}, \ldots, \kappa^{n_{l_i}}\}$ of $\{\kappa^1, \ldots, \kappa^n\}$ and a function $f_i : \mathbb{R}^{l_i} \to \mathbb{R}$ of class C^2 such that $Z^i = f_i(\kappa^{n_1}, \ldots, \kappa^{n_{l_i}})$.

Parts (i) and (ii) in Assumptions 8.5.7, combined with Lemma 8.4.1(i), yield the existence of a family of probability measures $\mathbb{P}^1, \ldots, \mathbb{P}^n$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ for some fixed $T > 0$, such that the process κ^i is a $(\mathbb{P}^i, \mathbb{F})$-martingale for every $i = 1, \ldots, n$. This in turn implies that κ^i is a $(\mathbb{P}, \mathbb{F})$-semimartingale. We deduce from part (iii) that the continuous martingale part of Z^i has the following integral representation

$$Z_t^{i,c} = Z_0^i + \sum_{j=1}^{l_i} \int_0^t \frac{\partial f_i}{\partial x_j}(\kappa_s^{n_1}, \ldots, \kappa_s^{n_{l_i}})\, d\kappa_s^{n_j,c}, \tag{8.92}$$

where $\kappa^{j,c}$ stands for the continuous martingale part of κ^j. To establish equality (8.92), it suffices to apply the Itô formula and use the properties of the stochastic integral.

We will argue that, under Assumptions 8.5.7, the semimartingale decomposition of κ^i can be uniquely specified under $\mathbb{P}$ by the choice of the initial values, the volatility processes and the driving martingale, which is, as usual, denoted by M. For the purpose of an explicit construction of the model for processes $\kappa^1, \ldots, \kappa^n$, we thus select an $\mathbb{R}^k$-valued $(\mathbb{P}, \mathbb{F})$-martingale $M = (M^1, \ldots, M^k)$ and we define the process κ^i under $\mathbb{P}^i$ as follows, for every $i = 1, \ldots, n$,

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \kappa_{s-}^i \sigma_s^i \cdot d\Psi^i(M)_s, \tag{8.93}$$

where σ^i is the $\mathbb{R}^k$-valued volatility process and the $(\mathbb{P}^i, \mathbb{F})$-martingale $\Psi^i(M)$ equals (cf. (8.24))

$$\Psi^i(M)_t = M_t - \int_{(0,t]} \frac{1}{Z_s^i}\, d[Z^{i,c}, M^c]_s - \sum_{0<s\leq t} \frac{1}{Z_s^i} \Delta Z_s^i \Delta M_s. \tag{8.94}$$

Proposition 8.5.7. *Under Assumptions 8.5.7, if the processes $\kappa^1, \ldots, \kappa^n$ satisfy (8.93)–(8.94) then for every $i = 1, \ldots, n$ the dynamics of κ^i are*

$$\begin{aligned} d\kappa_t^i = {} & \sum_{l=1}^k \kappa_{t-}^i \sigma_t^{i,l}\, dM_t^l - \frac{1}{f_i(\kappa_t^{n_1}, \ldots, \kappa_t^{n_{l_i}})} \sum_{j=1}^{l_i} \frac{\partial f_i}{\partial x_j}(\kappa_t^{n_1}, \ldots, \kappa_t^{n_{l_i}}) \sum_{l,m=1}^k \kappa_t^i \kappa_t^{n_j} \sigma_t^{i,l} \sigma_t^{n_j,m}\, d[M^{l,c}, M^{m,c}]_t \\ & - \frac{\kappa_{t-}^i}{Z_t^i} \Delta Z_t^i \sum_{l=1}^k \sigma_t^{i,l} \Delta M_t^l. \end{aligned} \tag{8.95}$$

Proof. In view of (8.93) and (8.94), it is clear that the continuous martingale part of κ^i can be expressed as follows

$$\kappa_t^{i,c} = \kappa_0^i + \int_{(0,t]} \kappa_s^i \sigma_s^i \cdot dM_s^c,$$

and thus we infer from (8.92) that $Z^{i,c}$ admits the following representation

$$Z_t^{i,c} = Z_0^i + \sum_{j=1}^{l_i} \int_{(0,t]} \frac{\partial f_i}{\partial x_j}(\kappa_s^{n_1}, \ldots, \kappa_s^{n_{l_i}}) \kappa_s^{n_j} \sigma_s^{n_j} \cdot dM_s^c. \tag{8.96}$$

This allows us to express the process $\Psi^i(M)$ in terms of M, the processes $\kappa^1, \ldots, \kappa^n$, their volatilities and the matrix $[M^c]$. To be more specific, we deduce from (8.94) and (8.96) that (note that $[M^{m,c}, M^c]$ is an $\mathbb{R}^k$-valued process)

$$\begin{aligned} \Psi^i(M)_t = M_t - \int_{(0,t]} \frac{1}{f_i(\kappa_s^{n_1}, \ldots, \kappa_s^{n_{l_i}})} \sum_{j=1}^{l_i} \frac{\partial f_i}{\partial x_j}(\kappa_s^{n_1}, \ldots, \kappa_s^{n_{l_i}}) \sum_{m=1}^{k} \kappa_s^{n_j} \sigma_s^{n_j,m} \, d[M^{m,c}, M^c]_s \\ - \sum_{0<s\le t} \frac{1}{Z_s^i} \Delta Z_s^i \Delta M_s, \end{aligned} \tag{8.97}$$

where $Z_s^i = f_i(\kappa_s^{n_1}, \ldots, \kappa_s^{n_{l_i}})$ and

$$\Delta Z_s^i = f_i(\kappa_s^{n_1}, \ldots, \kappa_s^{n_{l_i}}) - f_i(\kappa_{s-}^{n_1}, \ldots, \kappa_{s-}^{n_{l_i}}).$$

By combining (8.93) with (8.97), we arrive at formula (8.95). We thus conclude that, under Assumptions 8.5.7, the choice of the driving $(\mathbb{P}, \mathbb{F})$-martingale M and the volatility processes $\sigma^1, \ldots, \sigma^n$ in formula (8.93) completely specify the joint dynamics of $\kappa^1, \ldots, \kappa^n$ under $\mathbb{P}$. □

At the first glance, it is tempting to argue that the systems of SDEs derived in Propositions 8.5.1, 8.5.4, 8.5.5 and 8.5.6 can be seen as special cases of the system (8.95). Although formula (8.95) is fairly general, it should be stressed that this formula alone cannot be seen as a solution to the problem of modeling of forward swap rates and/or forward CDS spreads. Indeed, the crucial step in the top-down modeling approach is a correct specification of the functions f_i such that $Z^i = f_i(\kappa^{n_1}, \ldots, \kappa^{n_{l_i}})$. Obviously, this important step cannot be performed without a thorough analysis of a particular setup. Most likely, this will be done using the bottom-up approach in order to justify explicit formulae for the family of functions f_i.

In essence, the judicious choice of a considered family of forward swap rates and forward CDS spreads should ensure that the Radon-Nikodým densities of the corresponding martingale measures (that is, the processes Z^i appearing in Assumptions 8.5.7), which are defined as ratios of annuity deflated swap numéraires, should also admit unique representations in terms of the considered family of forward swap rates and/or forward CDS spreads, so that part (iii) in Assumptions 8.5.7 holds.

It is worth to stress that the corresponding functions f_i, as derived in Subsections 8.5.3 and 8.5.4 using the bottom-up approach, are not of class C^2 and thus the corresponding models require additional assumptions. We have circumvented this additional difficulty by working in Subsections 8.5.3 and 8.5.4 under an a priori assumption that all relevant processes belong to the class $\mathcal{S}_1$. Of course, this assumption should be validated a posteriori by showing that the systems of SDEs derived in these subsections have unique solutions such that the processes of interest are indeed in $\mathcal{S}_1$. These salient properties underpin in fact all particular cases of market models considered in this work. The corresponding general algebraic problem associated with the existence and uniqueness of functions f_i will be examined elsewhere. Let us finally observe that, without loss of generality, we may choose one of the processes Z^i to be equal identically to 1, so that for the corresponding martingale measure $\mathbb{P}^i$ for the process κ^i we have that $\mathbb{P}^i = \mathbb{P}$.

8.6 Concluding Remarks

We conclude this work by noting that the derivation of the joint dynamics of CDS spreads does not justify by itself that the model is suitable for the arbitrage pricing of credit derivatives. To show the viability of the model, it would be sufficient to demonstrate that the model of CDS spreads is arbitrage-free in some sense, for instance, that it can be supported by an associated arbitrage-free model for default-free and defaultable zero-coupon bonds.

Let us list some further issues that should be addressed in the context of top-down models:
(i) the existence and properties of a default time consistent with the dynamics of a given family of forward CDS spreads,
(ii) the positivity of processes modeling forward CDS spreads,
(iii) the positivity of prices of zero-coupon defaultable bonds implicitly defined by forward CDS spreads,
(iv) the monotonicity of prices of zero-coupon defaultable bonds with respect to maturity date,
(v) the positivity of other forward spreads that can be computed within a given model.

Last, but not least, one has also to analyze the calibration of models to market data and their applications to valuation and hedging of exotic credit derivatives, such as: constant maturity credit default swaps, Bermudan CDS options, etc. All these important issues are beyond the scope of this work and they are currently under study (for some preliminary results, we refer to [76]).

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