Continuity of a spatial derivative for a perturbed one-Laplace equation

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The model equation discussed in this talk

We talk about C^1-regulairity of weak solutions to, for example,

$$L_{b,\,p}u \coloneqq -b\Delta_1 u - \Delta_p u = f \in L^q(\Omega) \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \quad \text{where}$$

- Ω is a bounded n-dimensional domain with Lipschitz boundary.
- The unknown function $u\colon \Omega \to \mathbb{R}$ is in the class $W^{1,\,p}(\Omega)$.
- $\Delta_s u \coloneqq \operatorname{div}\left(|\nabla u|^{s-2}\nabla u\right)$ $(1 \le s < \infty)$ is the s-Laplacian.
- The given function $f\colon \Omega \to \mathbb{R}$ is in the class $L^q(\Omega)$.
- The constants & dimensions are assumed to be

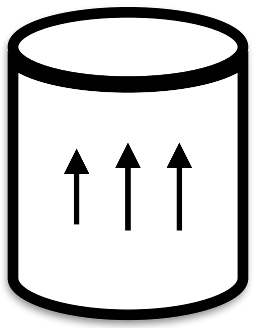
$$b \in (0,\,\infty),\ p \in (1,\,\infty),\ q \in (n,\,\infty],\ n \ge 2.$$

The equation

$$L_{b,p}u := -b\Delta_1 u - \Delta_p u = f \in L^q(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^n$$

is derived from the Euler–Lagrange equation

$$\frac{\delta \mathcal{F}}{\delta u} = 0 \quad \text{with} \quad \mathcal{F}(u) := b\int_\Omega |\nabla u| \, \mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \int_\Omega f \cdot u \, \mathrm{d}x.$$



This energy functional often appears in fields of

- Fluid mechanics (Bingham fluids) for $p = 2$.

  cf. Duvaut–Lions (Springer, Grundlehren series Vol. 219).

- Materials science (crystal surface growth) for $p = 3$.

  cf. Spohn (1993), Kohn (2012): fourth order problems.

Optimal Regularity

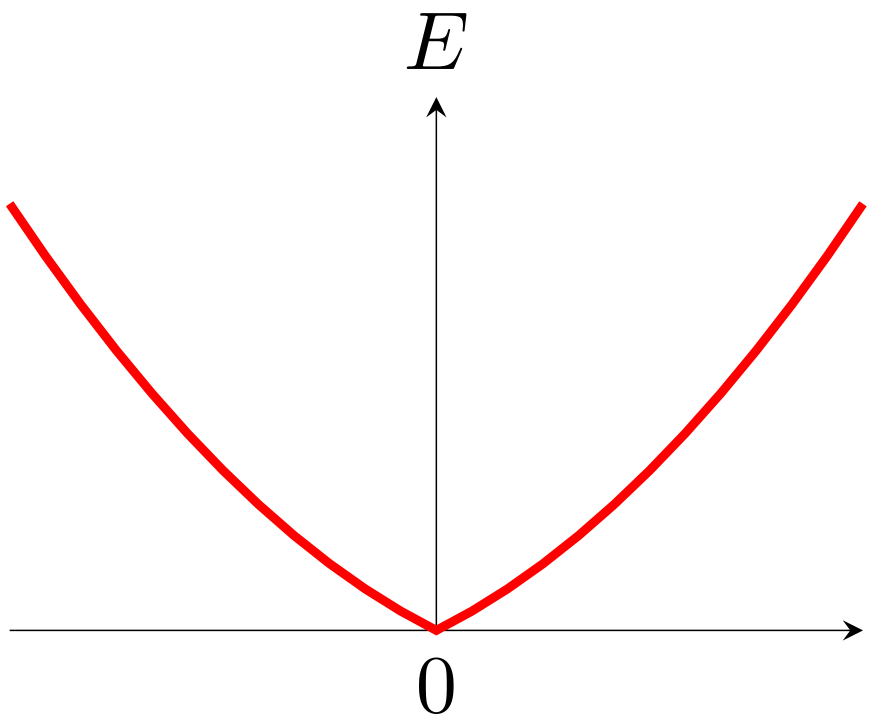
For a general energy density E (with smooth structure), we have

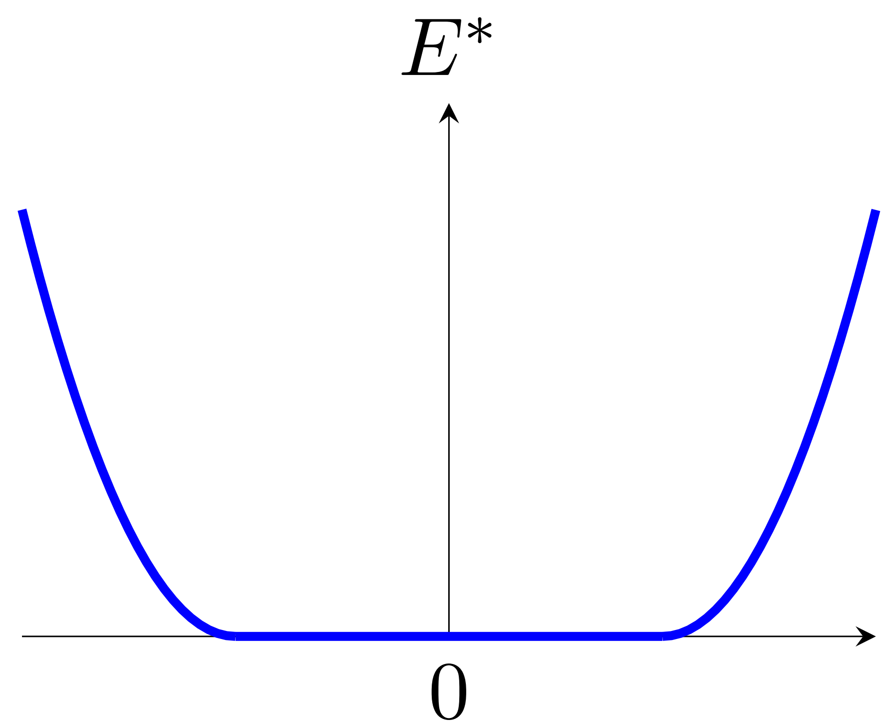
$$\nabla E(\nabla E^*(x)) = x, \quad E^*\text{: Fenchel dual.} \quad \text{(D)}$$

Hence $u := E^*$ is a solution to $-\operatorname{div}(\nabla E(\nabla u)) = -n$.

$\to C^{1,\alpha}$-regularity with $\alpha := \max\{\,1,\, (p-1)^{-1}\,\}$ is at most expectable.

$$E(z) = b|z| + \frac{1}{p}|z|^p. \qquad E^*(\zeta) = (p')^{-1}(|\zeta| - b)_+^{p'} \ \text{ with } p' := \frac{p}{p-1}.$$

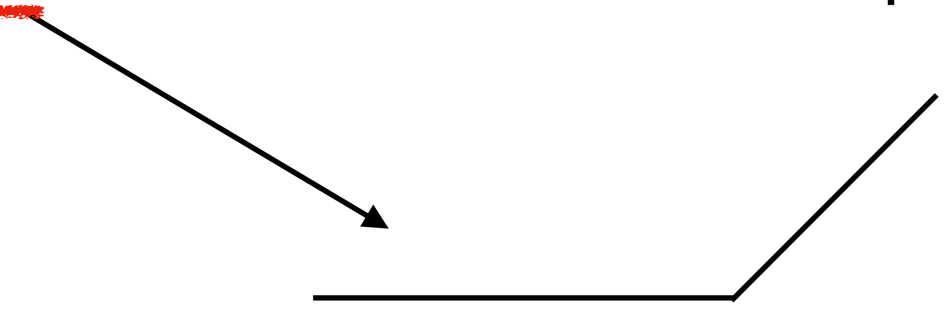




For p-Poisson problems (i.e., $b = 0$), $C^{1,\alpha}$-regularity is established by

- Uraltseva (1968), Uhlenbeck (1977), Evans (1982) for $p \in [2, \infty)$
- Lewis (1983), DiBenedetto (1983), Tolksdorff (1984) for $p \in (1, \infty)$

...and many experts. When $p = 1$, these results are not expectable.



The main result is

Theorem (T.; scalar (arXiv:2208.14640), system (Math. Ann., 2022))

A weak solution to $L_{b,p}u = f \in L^q$ with $q \in (n, \infty]$ is in $C^1(\Omega; \mathbb{R}^N)$.

cf. C^1-regularity when $N = 1$ & u: convex. (Y. Giga & T., ARMA, 2022)

Outline of Talk

1. Introduction and preliminary
   - Difficulty & Strategy
   - Comparison to related works on a very degenerate problem
2. A priori Hölder estimates for modulus-truncated gradients
   - Convergence of approximated solutions
   - Key Estimate
3. Generalization & Future Works
   - Generalizations
   - Future Works

Difficulty on regularity (1/3)

We fix $s \in [1, \infty)$ & consider a convex function

$$E_s(z) := \frac{1}{s}|z|^s \quad (z \in \mathbb{R}^n).$$

Then the Hessian matrix is given by,

$$\nabla^2 E_s(z_0) = (s-2)|z_0|^{s-4} z_0 \otimes z_0 + |z_0|^{s-2}\mathbf{1}_n \quad \text{for} \quad z_0 \in \mathbb{R}^n \setminus \{0\}.$$

In particular, its eigenvalues & corresponding eigenspaces are

$$\begin{cases} (s-1)|z_0|^{s-2} & \& \quad \mathbb{R}z_0 \quad (1 \text{ dimension}), \\ |z_0|^{s-2} & \& \quad (\mathbb{R}z_0)^\perp \quad (n-1 \text{ dimensions}). \end{cases}$$

Always 0 is an eigenvalue of $\nabla^2 E_1(z_0)$, even when $z_0 \in \mathbb{R}^n \setminus \{0\}$.

$\to$ Diffusivity of $\Delta_1 u$ degenerates in the direction ∇u, even when $\nabla u \neq 0$.

Difficulty on regularity (2/3)

We define

$$E(z) \coloneqq b|z| + \frac{1}{p}|z|^p \equiv bE_1(z) + E_p(z) \quad (z \in \mathbb{R}^n).$$

For each $z_0 \in \mathbb{R}^n \setminus \{0\}$, we compute

$$\begin{aligned}(\text{ER of } \nabla^2 E(z_0)) &\coloneqq \frac{(\text{the largest eigenvalue of } \nabla^2 E(z_0))}{(\text{the lowest eigenvalue of } \nabla^2 E(z_0))} \\ &= \frac{\max\{\, p-1,\, 1\,\} + b|z_0|^{1-p}}{\min\{\, p-1,\, 1\,\} + b \cdot 0}.\end{aligned}$$

This ER (Ellipticity Ratio) blows up as $|z_0| \to 0$.

In other words, **non-uniform ellipticity** appears as $\nabla u \to 0$.
cf. $(p,\, q)$-growth problems (non-uniform ellipticity as $|Du| \to \infty$).

Difficulty on regularity (3/3)

Going back to our Eq., written by

$$-\operatorname{div}(\nabla E(\nabla u)) = f.$$

$\downarrow$ Differentiate by x_j, then

$$-\operatorname{div}\left(\nabla^2 E(\nabla u)\nabla \partial_{x_j} u\right) = \partial_{x_j} f. \quad (\star)$$

Eq. $(\star)$ is **no longer** "uniformly elliptic", near $\{\nabla u = 0\}$ (facet).

$\to$ Growth estimate of $\partial_{x_j} u$ across $\{\nabla u = 0\}$ will be very hard.

Another Problem: Does $(\star)$ make sense in $W^{-1,\,2}$?

$\to$ We should relax $L_{b,\,p} = -b\Delta_1 - \Delta_p$ by *uniformly elliptic* operators.

Strategy (1/2)

Our Problem: $\quad -\operatorname{div}\left(\nabla^2 E(\nabla u)\nabla\partial_{x_j}u\right) = \partial_{x_j} f \quad (\star).$

Recall

$$\begin{aligned}(\mathrm{ER\ of\ } \nabla^2 E(\nabla u)) &\le C_p\left(1 + b|\nabla u|^{1-p}\right)\\ &\le C_p\left(1 + b\delta^{1-p}\right) < \infty\end{aligned}$$

when $|\nabla u| > \delta > 0$.

Roughly speaking,

1. Eq. $(\star)$ is "*locally* uniformly elliptic" in $\{\nabla u \neq 0\}$.
2. Its uniform ellipticity can be measured by $|\nabla u|$.

Strategy (2/2)

For each fixed $\delta \in (0,\,1)$ and $x_0 \in \Omega$, we would like to prove

$$\mathcal{G}_\delta(\nabla u) := (|\nabla u| - \delta)_+ \frac{\nabla u}{|\nabla u|} \in C^\alpha(B_\rho(x_0);\, \mathbb{R}^n)$$

for some $\alpha = \alpha(\delta,\, \mathrm{dist}(x_0,\, \partial\Omega)) \in (0,\,1)$.

Remark

The truncation mapping $\mathcal{G}_\delta$ satisfies $\sup\limits_{z\in\mathbb{R}^n} |\mathcal{G}_\delta(z) - z| \le \delta$.

This yields $\mathcal{G}_\delta(\nabla u) \to \nabla u$ uniformly in Ω. Thus, $\nabla u \in C^0$.

A very degenerate problem & its motivation

Our strategy can be already found in

$$-\operatorname{div}(\nabla E^*(\nabla v)) = f \in L^q(\Omega) \quad \text{with Neumann BC,}$$

where $E^*(\zeta) = \dfrac{1}{p'}(|\zeta| - b)_+^{p'}$ is the Fenchel dual of $E(z) = b|z| + \dfrac{|z|^p}{p}$.

This eq. is motivated by the duality formula

$$\nabla E^*(\nabla v) = \underset{\sigma \in L^p(\Omega;\mathbb{R}^n)}{\arg\min} \left\{ \int_\Omega E(\sigma)\mathrm{d}x \;\middle|\; \begin{array}{rcll} -\operatorname{div}\sigma & = & f & \text{in } \Omega, \\ \sigma \cdot \nu_{\partial\Omega} & = & 0 & \text{on } \partial\Omega, \end{array} \right\},$$

from congested traffic dynamics problems.

- Carlier–Jimenez–Santambrogio (2008): mathematical modeling.
- Brasco–Carlier–Santambrogio (2010): duality formula.

Regularity results on very degenerate problems

Theorem

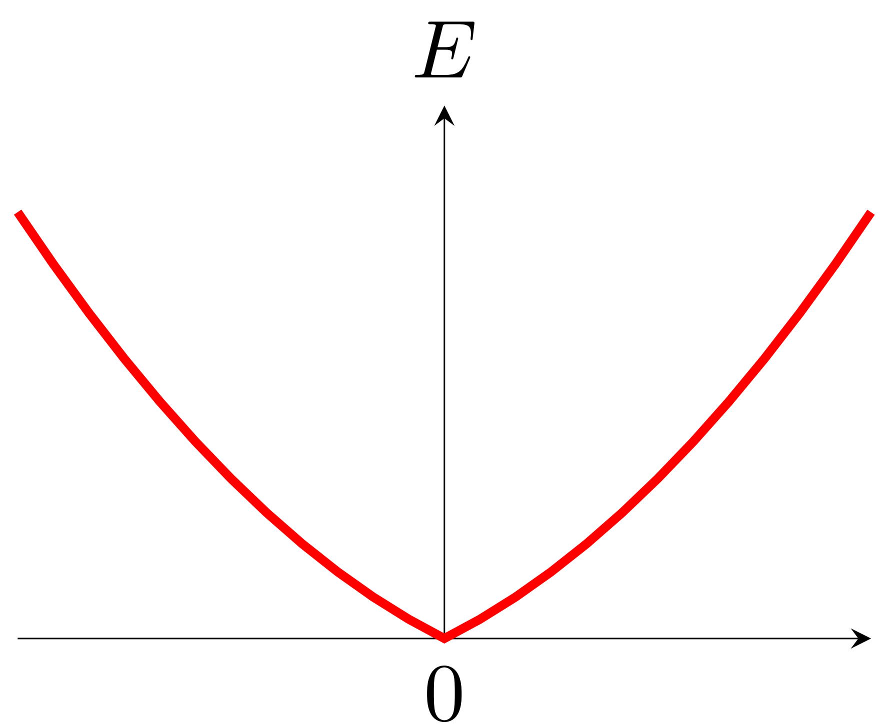
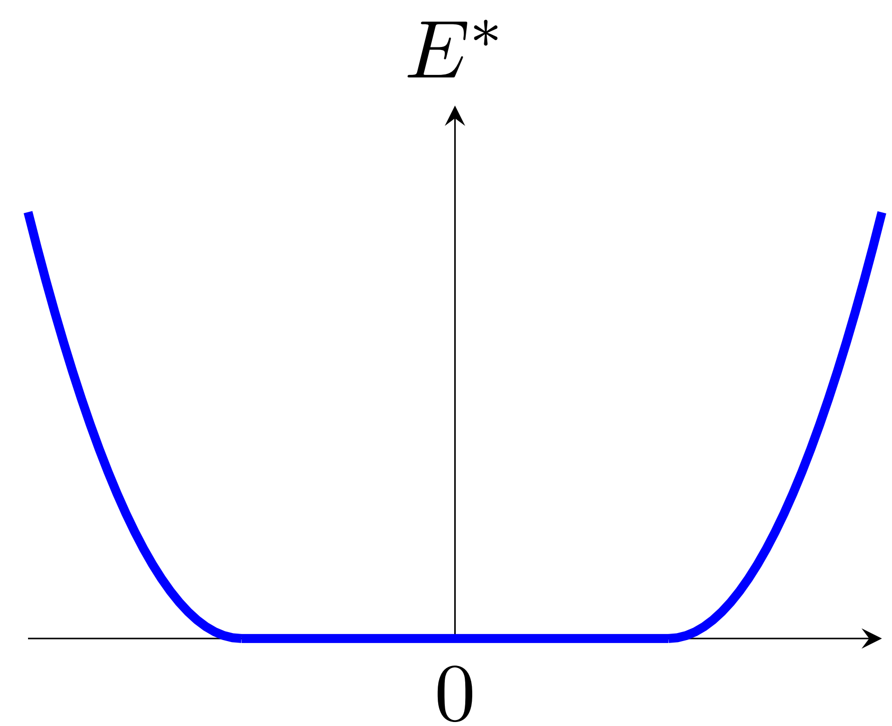
For each $\delta \in (0,\,1)$, $\mathcal{G}_{b+\delta}(\nabla v)\,(\delta > 0)$ is continuous.
In particular,

$$\mathcal{G}_b(\nabla v) = (|\nabla v| - b)_+ \frac{\nabla v}{|\nabla v|} \quad \& \quad \nabla E^*(\nabla v) \equiv |\mathcal{G}_b(\nabla v)|^{p'-2}\,\mathcal{G}_b(\nabla v)$$

are also continuous.

1. Santambrogio–Vespri (2010); $n = 2$ only.
2. Colombo–Figalli (2014); $n \geq 2$ & general E^*: convex.
3. Bögelein–Duzaar–Giova–Passarelli di Napoli (2022); $n \geq 2$ & system.
   ← De Giorgi's truncation & Freezing coefficient method.

Structures on Ellipticity Ratio (ER)

Eq.	$-\operatorname{div}(\nabla E(\nabla u)) = f$	$-\operatorname{div}(\nabla E^{*}(\nabla v)) = f$
Diffusivity	very singular	very degenerate
Density	$E(z) = b\vert z\vert + \frac{1}{p}\vert z\vert^{p}$	$E^{*}(\zeta) = \frac{1}{p'}(\vert\zeta\vert - b)_{+}^{p'}$
		
ER	$C_{b,\,p}(1 + \vert\nabla u\vert^{1-p})$	$C_{p}(1 + (\vert\nabla v\vert - b)_{+}^{-1})$
Regularity	$\mathcal{G}_{0}(\nabla u) = \nabla u \in C^{0}$	$\mathcal{G}_{b}(\nabla v) \in C^{0}$

Definition of weak solutions

Definition (T., Weak solution in the distributional sense)

A function $u \in W^{1,p}(\Omega)$ is called a weak solution to $L_{b,p}u = f$ in Ω, when there exists $Z \in L^{\infty}(\Omega; \mathbb{R}^n)$ such that

- $\displaystyle\int_{\Omega} \left(Z + |\nabla u|^{p-2}\nabla u\right) \cdot \nabla\varphi \, \mathrm{d}x = \int_{\Omega} f\varphi \, \mathrm{d}x$ for all $\varphi \in W_0^{1,p}(\Omega)$,
- $Z(x) \in \partial|\cdot|(\nabla u(x))$ for a.e. $x \in \Omega$.

The subdifferential of $|\cdot| \colon \mathbb{R}^n \to [0, \infty)$ is given by

$$\mathbb{R}^n \supset \partial|\cdot|(z_0) = \begin{cases} \{\, |z_0|^{-1} z_0 \,\} & (z_0 \neq 0), \\ \{\zeta \in \mathbb{R}^n \mid |\zeta| \leq 1\} & (z_0 = 0). \end{cases}$$

Note: $|\cdot|$ is not differentiable at the origin.

Approximation of $L_{b,\,p}u$

- We approximate $L_{b,\,p}u = -b\Delta_1 u - \Delta_p u$ by
$$L_{b,\,p}^{\varepsilon}u_\varepsilon := -\operatorname{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}}\right) - \operatorname{div}\left(\left(\varepsilon^2 + |\nabla u_\varepsilon|^2\right)^{p/2-1}\nabla u_\varepsilon\right)$$
for an approximation parameter $\varepsilon \in (0,\,1)$.

- This naturally appears when relaxing $E = b|z| + \dfrac{|z|^p}{p}$ by
$$E_\varepsilon(z) := b\sqrt{\varepsilon^2 + |z|^2} + \frac{1}{p}\left(\varepsilon^2 + |z|^2\right)^{p/2}.$$

- The operator $L_{b,\,p}^{\varepsilon}$ is uniformly elliptic, in the sense that
$$(\text{ER of } \nabla^2 E_\varepsilon(z_0)) \le 1+\left(\varepsilon^2 + |z_0|^2\right)^{(1-p)/2} \le 1+\varepsilon^{1-p} < \infty \quad \forall z_0 \in \mathbb{R}^n.$$
$\rightarrow$ Standard elliptic arguments (difference quotient etc.) are useful.

Convergence (1/2)

We introduce $\varepsilon \in (0,\,1)$ & consider

$$(\mathsf{E}_\varepsilon) \quad L^{\varepsilon}_{b,\,p} u_\varepsilon \equiv -\operatorname{div}\left(\nabla E_\varepsilon(\nabla u_\varepsilon)\right) = f_\varepsilon \quad \text{in} \quad \Omega,$$

where $f_\varepsilon \rightharpoonup f$ in $\sigma(L^q,\, L^{q'})$. In particular, we may let $f_\varepsilon \in C^\infty$.

Proposition

Let $u \in W^{1,\,p}(\Omega)$ be a weak solution to (E) & $\varepsilon \in (0,\,1)$. Consider the unique function $u_\varepsilon \in u + W^{1,\,p}_0(\Omega)$ that satisfies (E_ε) in the weak sense,

i.e., $$\int_\Omega \left(\frac{b \nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} + \left(\varepsilon^2 + |\nabla u_\varepsilon|^2\right)^{(p-2)/2} \nabla u_\varepsilon \right) \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f_\varepsilon \varphi \, \mathrm{d}x$$

holds for all $\varphi \in W^{1,\,p}_0(\Omega)$. Then $u_\varepsilon \to u$ in $W^{1,\,p}(\Omega)$ (up to a sub-seq.).

Keypoint: Δ_p gives *quantitative* monotonicity estimates.

Convergence (2/2)

In particular, a weak solution u to $L_{b,\,p}u = f$ can be approximated by

$$(\mathrm{D}_\varepsilon)\begin{cases} L^{\varepsilon}_{b,\,p}u_\varepsilon & = & f_\varepsilon & \text{in} & \Omega, \\ u_\varepsilon & = & u & \text{on} & \partial\Omega. \end{cases}$$

Unique existence of weak solution of (D_ε) is easy. In fact, we have

$$u_\varepsilon = \arg\,\min \left\{ \int_\Omega \left[E_\varepsilon(\nabla v) - f_\varepsilon v \right] \mathrm{d}x \,\middle|\, v \in u + W_0^{1,\,p}(\Omega) \right\}.$$

Some Remarks

Our Theorem will be reduced to a priori estimates for

$$L^{\varepsilon}_{b,\,p} u_\varepsilon \equiv -\operatorname{div}(\nabla E_\varepsilon(\nabla u_\varepsilon)) = f_\varepsilon.$$

$\downarrow$ Differentiate by x_j, then

$$-\operatorname{div}\left(\nabla^2 E_\varepsilon(\nabla u_\varepsilon)\nabla\partial_{x_j} u_\varepsilon\right) = \partial_{x_j} f_\varepsilon. \quad (*)$$

1. $u_\varepsilon \in W^{2,\,2}_{\mathrm{loc}} \cap W^{1,\,\infty}_{\mathrm{loc}}$ is expectable (cf. Giusti, World Scientific).
   In particular, Eq. $(*)$ makes sense locally in $W^{-1,\,2}$.
2. ER (Ellipticity Ratio) should be measured

   $$\text{by} \quad V_\varepsilon := \sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}, \quad \text{not by} \quad |\nabla u_\varepsilon|.$$

   Note: $c(p)V_\varepsilon^{p-2}\mathbf{1}_n \leqslant \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \leqslant \left(C(p)V_\varepsilon^{p-2} + bV_\varepsilon^{-1}\right)\mathbf{1}_n$, so that

   $$(\text{ER of } \nabla^2 E_\varepsilon(\nabla u_\varepsilon)) \le C_p\left(1 + bV_\varepsilon^{1-p}\right) \le C_{b,\,p,\,\varepsilon} < \infty.$$

Key estimate: A priori Hölder bounds

Proposition (T., A priori Hölder bounds for $\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon)$)

Let $\delta \in (0,\,1)$ & $\varepsilon \in (0,\,\delta/8)$. Then for each $x_0 \in \Omega$, we have

$$\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon) := \left(\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2} - 2\delta\right)_+ \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \in C^\alpha(B_\rho(x_0);\,\mathbb{R}^n),$$

where the radius $\rho > 0$ and the Hölder exponent $\alpha \in (0,\,1)$

- may depend on δ and $d_0 = \mathrm{dist}(x_0,\,\partial\Omega)$,
- but are independent of ε.

Moreover, we have

$$|\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon(x_1)) - \mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon(x_2))| \le C|x_1 - x_2|^\alpha \quad \text{for all } x_1,\,x_2 \in B_\rho(x_0)$$

with $C = C(\delta,\,d_0) \in (0,\,\infty)$ independent of ε.

Note: $\mathcal{G}_{2\delta}(\nabla u) \in C^\alpha(B_\rho(x_0);\,\mathbb{R}^n)$ follows from the Arzelà–Ascoli theorem.

Sketch of Hölder a priori estimates

Aim: $\forall x_0 \in B$, $\exists \Gamma_{2\delta,\,\varepsilon}(x_0) := \lim_{r\to 0} \left(\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon)\right)_{x_0,\,r} \in \mathbb{R}^n$ satisfies

$$\fint_{B_r(x_0)} \left|\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon) - \Gamma_{2\delta,\,\varepsilon}(x_0)\right|^2 \,\mathrm{d}x \lesssim \mu^2 \left(\frac{r}{\rho}\right)^{2\alpha}.$$

Our analysis depends on whether a modulus V_ε is

non-degenerate	$\to$	Freezing coefficient arguments,
degenerate	$\to$	De Giorgi's truncation,

which can be judged by <u>measuring super-levelsets</u>.

cf. E. DiBenedetto, *"Degenerate Parabolic Equations"* (Springer).

Generalizations for everywhere C^1-regularity

Scalar case ($N = 1$). arXiv:2208.14640 (preprint)

- Both convex functions $E_1 = b|z|$ & $E_p = |z|^p/p$ are generalized.
- Approximate $E = E_1 + E_p$ by convoluting by Friedrichs' mollifiers.
- This scheme works as long as E_1 is positively one-homogeneous,

$$\text{i.e., } E_1(kz) = kE_1(z) \quad \forall z \in \mathbb{R}^n \quad \& \quad \forall k \in (0,\, \infty).$$

Vector case ($N \geq 2$). Math. Ann. (2022)

- We should impose $E(z) = g(|z|^2)$ (Uhlenbeck structure).
- This symmetry is used to deduce weak forms of $V_\varepsilon = \sqrt{\varepsilon^2 + |Du_\varepsilon|^2}$.
- In particular, one-homogeneous E_1 should be $E_1(z) = b|z|$.
- E is approximated by $E_\varepsilon(z) := g(\varepsilon^2 + |z|^2)$.

For both cases, E is required to be (at least) C^2 outside the origin.

Future Works (1/2)

1. Growth estimates of ∇u across a facet $\{\nabla u = 0\}$ (less is known).

2. Other analysis (e.g., localization) for *non-divergence* problems.

   cf. Evans & Savin (2008) : $C^{1,\alpha}$-regularity for $\Delta_\infty u = 0$ with $n = 2$.

   cf. De Silva–Savin (2010), Mooney (2020):
   $\to$ C^1-regularity for minimizers of some strictly convex energy.

3. Parabolic problems (in preparation). Consider

$$\partial_t u - \Delta_1 u - \Delta_p u = f(x, t) \in L^q(\Omega_T; \mathbb{R}^N) \quad \text{in} \quad \Omega_T := \Omega \times (0, T) \quad \text{(P)}$$

   Question: Is a spatial gradient $Du \in L^p(\Omega_T; \mathbb{R}^{Nn})$ continuous when

$$\frac{2n}{n+2} < p < \infty, \quad \text{and} \quad n + 2 < q \le \infty\,?$$

   Answer: Yes, but some restrictions are (technically) required.

Parabolic case

We let $N = 1$,

$$\frac{2n}{n+2} < p < \infty, \quad n+2 < q < \infty, \quad \& \quad \frac{1}{p} + \frac{1}{q} \le 1. \quad \text{(A)}$$

$$X^p(0,\, T;\, \Omega) := \left\{ u \in L^p(0,\, T;\, W^{1,\,p}(\Omega)) \,\middle|\, \partial_t u \in L^{p'}(0,\, T;\, W^{-1,\,p'}(\Omega)) \right\},$$

$$X_0^p(0,\, T;\, \Omega) := \left\{ u \in L^p(0,\, T;\, W_0^{1,\,p}(\Omega)) \,\middle|\, \partial_t u \in L^{p'}(0,\, T;\, W^{-1,\,p'}(\Omega)) \right\}.$$

Definition

A function $u \in X^p(0,\, T;\, \Omega)$ is said to be a weak solution to (P), when $\exists Z \in L^\infty(\Omega_T)$ s.t.

- $Z(x,\, t) \in \partial|\cdot|(\nabla u(x,\, t))$ for a.e. $(x,\, t) \in \Omega_T$.
- $\forall \varphi \in X_0^p(0,\, T;\, \Omega)$ with $\varphi|_{t=0} = \varphi|_{t=T} = 0$ in $L^2(\Omega)$, there holds

$$-\iint_{\Omega_T} u \partial_t \varphi \, \mathrm{d}x\mathrm{d}t + \iint_{\Omega_T} \left\langle Z + |\nabla u|^{p-2} \nabla u \,\middle|\, \nabla \varphi \right\rangle \mathrm{d}x\mathrm{d}t = \iint_{\Omega_T} f \varphi \, \mathrm{d}x\mathrm{d}t.$$

3 Remarks on the conditions (A);

1. Gelfand triple $W_0^{1,p}(\Omega) \overset{\text{cpt}}{\hookrightarrow} L^2(\Omega) \overset{\text{conti}}{\hookrightarrow} W^{-1,p'}(\Omega)$ by $p > \dfrac{2n}{n+2}$.

   $\to X_0^p(0, T; \Omega) \subset C([0, T]; L^2(\Omega)) \quad \& \quad X_0^p(0, T; \Omega) \overset{\text{cpt}}{\hookrightarrow} L^p(0, T; L^2(\Omega))$.

2. Existence theory for $\partial_t u - \Delta_p u = f$ in $L^{p'}(0, T; W^{-1,p'}(\Omega))$

   $\to$ J.-L. Lions (1969) & Showalter (1997) for $p > \dfrac{2n}{n+2}$.

3. The condition $1/p + 1/q \leq 1$ allows us to use

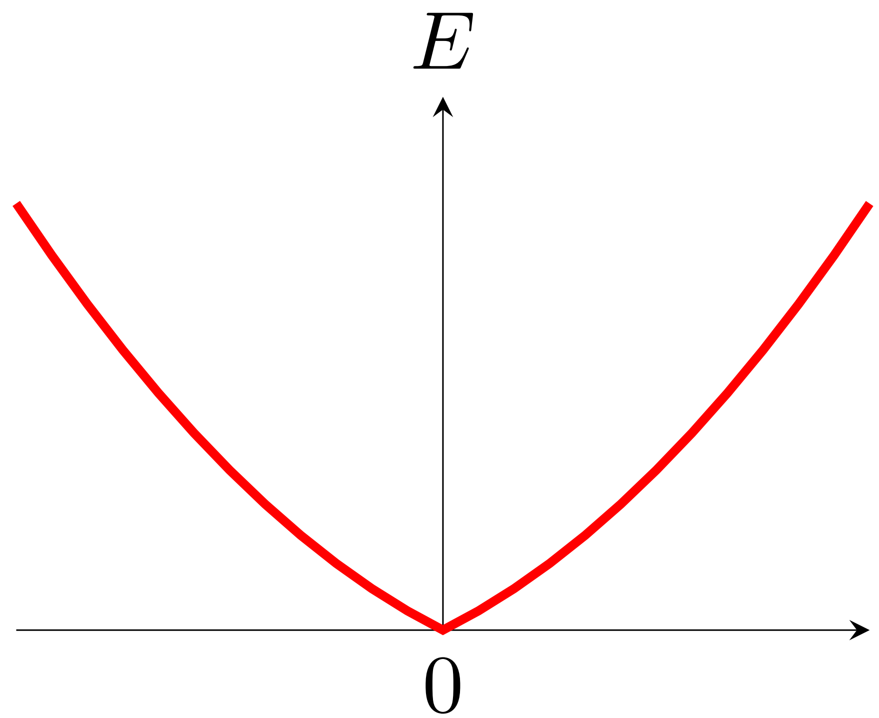
$$L^q(\Omega_T) = L^q(0, T; L^q(\Omega)) \overset{\text{conti}}{\hookrightarrow} L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

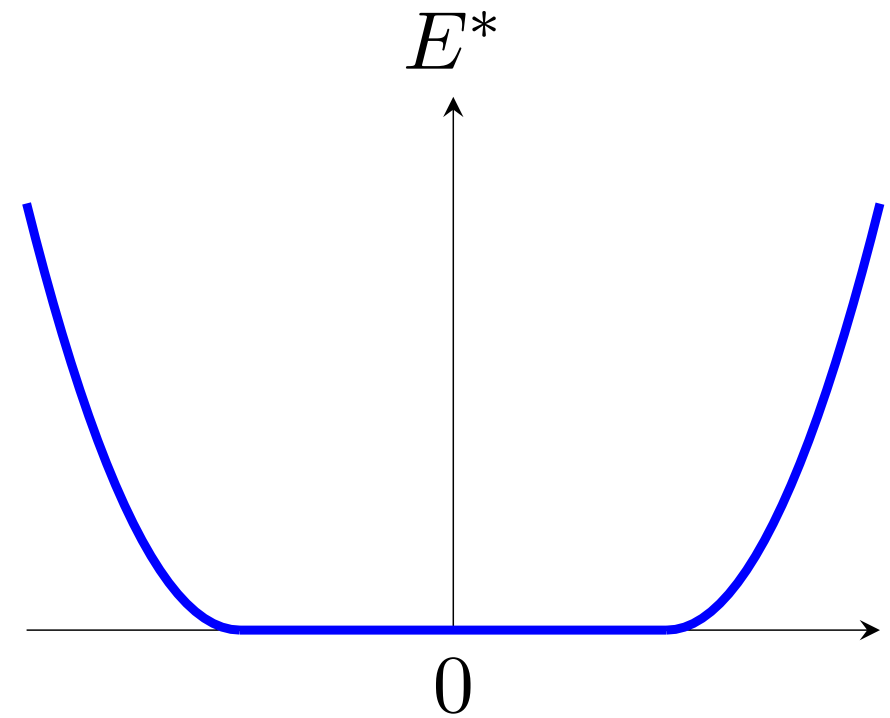
Theorem (T., (in preparation))

Let u be a weak solution to (P) with (A). Then, $\nabla u \in C^0(\Omega_T; \mathbb{R}^n)$.

Future Works (2/2): Duality & Regularity

Diffusivity — very singular — very degenerate





Elliptic Regularity — $\mathcal{G}_0(\nabla u) = \nabla u \in C^0$ — $\mathcal{G}_b(\nabla v) \in C^0$

Q. Parabolic Regularity & Partial Regularity ($N \geq 2$ & non-Uhlenbeck)?

cf. Ambrossio–Passarelli di Napoli (arXiv:2204.05966),
Gentile–Passarelli di Napoli (arXiv:2301.11795),

$\rightarrow$ Parabolic regularity for degenerate cases ($\mathcal{G}_b(\nabla v) \in C^0$ remains open).