On the L_p dual Minkowski problem

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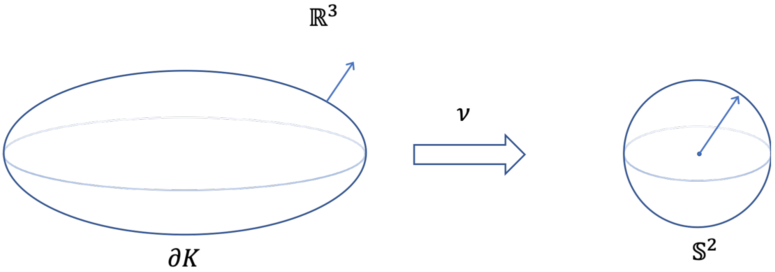
(joint work with K. Choi and M. Kim)



- $K \mapsto \mu_K$
- For example, surface area or cone volume.
- Can we characterize geometric measures μ_K?
  (Equivalently, what is the image of the mapping $K \mapsto \mu_K$)

Surface area measure

- Let K be a convex body in $\mathbb{R}^{n+1}$ (compact convex set with nonempty interior), and let $\nu_K : \partial K \to \mathbb{S}^n$ be the outward unit normal vector.
- Any convex body defines the so called *surface area measure* on $\mathbb{S}^n$: The surface area measure $S(K, \cdot)$ of K is defined on a Borel set $\omega \subset \mathbb{S}^n$ by

$$S(K, \omega) = |\nu_K^{-1}(\omega)|,$$

  where $|\cdot|$ denotes the surface area.
- Total measure: $S(K, \mathbb{S}^n) = |\nu_K^{-1}(\mathbb{S}^n)| = |\partial K|$.

- Observation: if μ is a surface area measure, then
  1. Surface area measure has centroid at origin:
  $$\int_{\mathbb{S}^n} z \, \mathrm{d}\mu(z) = \int_{\partial K} \nu(x) d\mathcal{H}^n(x) = o.$$
  2. Surface area measure is not concentrated on a great subsphere:
  $$\mu(E) \neq \mu(\mathbb{S}^n) \quad \text{for all great subsphere } E \subset \mathbb{S}^n.$$
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- Can we characterize the surface area measure?
- **Minkowski problem:** For a given nonzero finite Borel measure μ on $\mathbb{S}^n$, what are the necessary and sufficient conditions for $\mu = S(K, \cdot)$ for some convex body K? (Minkowski, 1903)
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- **Minkowski problem:** For a given nonzero finite Borel measure μ on $\mathbb{S}^n$, what are the necessary and sufficient conditions for $\mu = S(K, \cdot)$ for some convex body K? (Minkowski, 1903)
- Minkowski problem is completely solved by Minkowski (discrete case) and Alexandrov (general case).
- $\mu = S(K, \cdot)$ for a convex body $K \iff$ 1. and 2. hold for μ.

- In smooth category ($\mu = f \mathrm{d}\sigma_{\mathbb{S}^n}$), the Minkowski problem becomes solving the following Monge–Ampère type PDE on $\mathbb{S}^n$:

$$\det(\nabla_i \nabla_j u + u\delta_{ij}) = \frac{1}{\mathcal{K}} = f \quad \text{on } \mathbb{S}^n,$$

where $\mathcal{K}$ is the Gauss curvature and u is the support function of K.

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  where $\mathcal{K}$ is the Gauss curvature and u is the support function of K.
- Uniqueness: The convex body is unique up to translation.
- Regularity: If $f \in C^\alpha$, then $\partial K \in C^{2,\alpha}$. ($C^\infty$ regularity by Pogorelov, Nirenberg, Cheng–Yau, and $C^{2,\alpha}$ regularity by Caffarelli)
- Therefore, the surface area measures are characterized by 1. and 2. In which case, the solution convex body is well understood.

Variational point of view

- Let $h_L : \mathbb{S}^n \to \mathbb{R}$ be the support function of L defined by

$$h_L(z) = \max\{z \cdot x : x \in L\},$$

and let $K + L = \{x + y : x \in K, y \in L\}$ be the Minkowski sum.

- Aleksandrov variational formula:

$$\left.\frac{\mathrm{d}\,\mathrm{Vol}(K + tL)}{\mathrm{d}t}\right|_{t=0^+} = \int_{\mathbb{S}^n} h_L(z)\,\mathrm{d}S(K, z)$$

- Firey's p-linear combination $K +_p L$ of K and L ($p \geq 1$):

$$h_{K+_pL} = (h_K^p + h_L^p)^{1/p}, \quad h_{t\cdot_p L} = t^{1/p} h_K$$

- There exists a Borel measure $S_p(K, \cdot)$ on $\mathbb{S}^n$ such that

$$\left.\frac{\mathrm{d}\,\mathrm{Vol}(K +_p t \cdot_p L)}{\mathrm{d}t}\right|_{t=0^+} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h_L^p(z)\,\mathrm{d}S_p(K, z).$$

L_p surface area measure

- The measure $S_p(K,\cdot)$ is called as the L_p surface area measure.
- It turns out that for $p \geq 1$,

$$\mathrm{d}S_p(K,\cdot) = h_K^{1-p}\mathrm{d}S(K,\cdot).$$

- The L_p surface area measure can be defined for all $p \in \mathbb{R}$ through the relation above.
- L_p **Minkowski problem:** For a given nonzero finite Borel measure μ on $\mathbb{S}^n$, what are the necessary and sufficient conditions for $\mu = S_p(K,\cdot)$ for some convex body K? (Lutwak '93)
- PDE: for a density function f,

$$\det(\nabla_i\nabla_j u + u\delta_{ij}) = \frac{1}{\mathcal{K}} = u^{p-1}f \quad \text{on } \mathbb{S}^n.$$

- Examples: classical case ($p = 1$), logarithmic case ($p = 0$), affine case ($p = -n-1$),

Dual curvature measure

- Let r_K be the radial function of K defined by

$$r_K(\xi) = \max\{\lambda : \lambda\xi \in K\}.$$

- The q-th dual volume of K is

$$\widetilde{\mathrm{Vol}}_q(K) = \frac{1}{n+1}\int_{\mathbb{S}^n} r_K^q(\xi)\mathrm{d}\xi.$$

- The q-th dual curvature measure is determined by $(q \neq 0)$

$$\left.\frac{\mathrm{d}\widetilde{\mathrm{Vol}}_q(K+tL)}{\mathrm{d}t}\right|_{t=0^+} = q\int_{\mathbb{S}^{n-1}} h_L h_K^{-1}\,\mathrm{d}\,\tilde{C}_q(K,\cdot).$$

- For any $\omega \subset \mathbb{S}^n$,

$$\widetilde{C}_q(K,\omega) = \int_{\mathcal{A}^*(\omega)} r_K^q(\xi)\mathrm{d}\sigma_{\mathbb{S}^n}(\xi),$$

where $\mathcal{A}^*$ is the reverse radial Gauss mapping defined as

$$\mathcal{A}^*(\omega) = \{\xi \in \mathbb{S}^n : \nu_K(r_K(\xi)\xi) \in \omega\}.$$

Dual Minkowski problem

- **Dual Minkowski problem:** For a given nonzero finite Borel measure μ on $\mathbb{S}^n$, what are the necessary and sufficient conditions for $\mu = \widetilde{C}_q(K, \cdot)$ for some convex body K? (Huang–Lutwak–Yang–Zhang '16).
- PDE: for $r = \sqrt{u^2 + |\nabla u|^2}$,

$$\det(\nabla_i \nabla_j u + u\delta_{ij}) = \frac{r^{n+1-q}}{u} f \quad \text{on } \mathbb{S}^n,$$

- Examples: the logarithmic Minkowski problem ($q = n+1$) and the Alexandrov problem ($q = 0$)
- The logarithmic case appears not only in the L_p Minkowski problem but also in the dual Minkowski problem.
- What is next?

L_p Dual Minkowski problem

- The L_p dual curvature measure $\widetilde{C}_{p,q}(K,\cdot)$ is produced by

$$\left.\frac{\mathrm{d}\widetilde{\mathrm{Vol}}_q(K+_p t\cdot_p L)}{\mathrm{d}t}\right|_{t=0^+} = q\int_{\mathbb{S}^n} h_L^p(z)\,\mathrm{d}\tilde{C}_{p,q}(K,z).$$

- L_p **Dual Minkowski problem:** For a given nonzero finite Borel measure μ on $\mathbb{S}^n$, what are the necessary and sufficient conditions for $\mu = \widetilde{C}_{p,q}(K,\cdot)$ for some convex body K? (Lutwak–Yang–Zhang '18).
- Relation with the dual curvature measure is given by

$$\widetilde{C}_{p,q}(K,\cdot) = h_K^{-p}\widetilde{C}_q(K,\cdot)$$

- PDE: for $r = \sqrt{u^2+|\nabla u|^2}$,

$$\det(\nabla_i\nabla_j u + u\delta_{ij}) = \frac{r^{n+1-q}}{u^{1-p}}f \quad \text{on } \mathbb{S}^n$$

- Examples: the L_p Minkowski problem ($q=n+1$), the dual Minkowski problem ($p=0$).

Logarithmic Minkowski problem ($p = 0, q = n + 1$)

- We first consider L_p Minkowski problem.
- In particular, $p = 0$, corresponds to the logarithmic Minkowski problem. This is related to the cone volume:

$$\frac{1}{n+1} \mathrm{d}S_0(K, \cdot) = \frac{1}{n+1} h_K \mathrm{d}S(K, \cdot), \quad \frac{1}{n+1} S_0(K, \mathbb{S}^n) = \mathrm{Vol}(K)$$

- In 2013, Böröczky–Lutwak–Yang–Zhang solved the logarithmic case under even assumption ($\mu(E) = \mu(-E)$):

$$\mu = S_0(K, \cdot) \iff 1. \quad \frac{\mu(\xi \cap \mathbb{S}^n)}{\mu(\mathbb{S}^n)} \leq \frac{\dim(\xi)}{n+1}, \quad \xi \leq \mathbb{R}^{n+1}$$

2. some extra condition when equality holds

- Non-symmetric case is open.
- For other $p \neq 0, 1$, some sufficient conditions have been provided, but the L_p Minkowski problem is still open for symmetric or non-symmetric, except for the lower dimensional case ($n = 1$).
- Finding necessary and sufficient conditions are widely open.

Measure with density

- Recall the PDE: for a density function f,

$$u^{1-p}\det(\nabla_i\nabla_j u + u\delta_{ij}) = \frac{u^{1-p}}{\mathcal{K}} = f \quad \text{on } \mathbb{S}^n$$

- Existence of solutions is guaranteed for sufficiently smooth, positive f. We mainly focus on the uniqueness and regularity (or existence of regular solutions).
- Soliton of (anisotropic) α-Gauss curvature flow through the relation $\alpha = 1/(1-p)$.
- C^0 estimate or diameter estimate is important.
  Blaschke selection theorem (compactness): Let $\{K_n\}$ be a sequence of convex bodies contained in fixed bounded set. Then there is convex body K such that (up to subsequence)

$$K_i \to K \quad \text{in Hausdorff distance.}$$

- Positive lower bound on u is crucial for regularity. (whether the origin lies in the interior or not)

Overview for various range of p

- $p > n+1$: Existence, uniqueness, regularity
  At the maximum point of u, it follows from the PDE that

$$u_{\max}^{1-p+n} \geq f_{\min}, \quad u_{\max} \leq \frac{1}{f_{\min}^{1/(p-n-1)}}, \quad u_{\min} \geq \frac{1}{f_{\max}^{1/(p-n-1)}}.$$

- $1 < p < n+1$: Example of a convex body with the origin on its boundary. Weak solution and uniqueness. Regularity for even case.
- $-n-1 < p < 0$: No diameter estimate, but existence of weak solutions. No uniqueness. If $-n-1 < p \leq -n+1$, then solution is positive.
- $p < -n-1$: Existence (Guang–Li–Wang 22, arxiv) and ...?
- $p = 0$: If $n = 1$, then diameter estimate and positiveness of solutions hold (Chen–Li 18). Therefore existence, uniqueness, regularity follows when $n = 1$. If $n = 2$, then diameter estimate holds (Chen–Feng–Liu 22, arxiv). Diameter estimate for $n \geq 3$ is open.
- $0 < p < 1$: Does the diameter estimate hold?

Result 1. Diameter estimate when $n = 1$

Theorem (Kim–L. 22, arxiv)

Let $p \in (0,1)$, and let f be a bounded, positive function on $\mathbb{S}^1$. If K is a convex body such that

$$h_K^{1-p}((h_K)_{\theta\theta} + h_K) = f \quad on\ \mathbb{S}^1, \tag{*}$$

then $\|h_K\|_{L^\infty} \leq C$ for some $C = C(p, \Lambda)$.

Remark 1. Diameter estimate for $n \geq 2$ is open.
Remark 2. If $p = 0$ or $p = 1$, then the LHS of (*) is cone volume or surface area measure, respectively. In these case, one can use monotone property of volume or surface area (of convex bodies). However, $S_p(K, \cdot)$ does not have such monotone properties.

Idea of proof.

1. Key estimate on the L_p surface area:

$$S_p(K, \mathbb{S}^1) \simeq \mathrm{Vol}(K)^{1-p} |\partial K|^p (\simeq C)$$

2. Consider a sequence of convex bodies $\{K_i\}$ with $\mathrm{diam}(K_i) \to \infty$.
3. Case I: The origin lies near the tip. Near the tip (denoted by ω),

$$\int_\omega f \simeq C, \quad S_p(K, \omega) \leq \mathrm{Vol}^{1-p}\,\mathrm{Area}^p \lesssim \epsilon\,\mathrm{Vol}^{1-p}(K)|\partial K|^p \lesssim \epsilon S_p(K, \mathbb{S}^1) \lesssim \epsilon.$$

4. Case II: the origin lies far from tips. On the complement of neighborhoods of tips (denoted by ω),

$$\int_\omega f \simeq 0, \quad \text{but} \quad S_p(K, \omega) \gtrsim S_p(K, \mathbb{S}^1) \gtrsim 1.$$

□

Uniqueness

- The L_p Brunn–Minkowski inequality holds for $p \geq 1$:

$$\mathrm{Vol}((1-t)\cdot_p K +_p t \cdot_p L) \geq \mathrm{Vol}(K)^{1-t}\,\mathrm{Vol}(L)^t$$

  This will give the uniqueness for $p \geq 1$.
- For $p < 1$, there exists f that admits more than two solutions.
- When $f \equiv 1$, the uniqueness for $-n-1 < p < 1$ has been established by Chow '85 ($p = -n+1$), Andrews '99 ($p = 0, n = 2$), Brendle–Choi–Daskalopoulos '17
  c.f. Guan–Ni, Andrews–Guan–Ni, Kim–Lee for convergence of flow.
- More generally, the uniqueness holds when f is even (Bryan–Ivaki–Scheuer '19).

Corollary

Let $p \in (0,1)$ and $f \in C^\alpha(\mathbb{S}^1)$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(p) > 0$ such that if $\|f-1\|_{C^\alpha(\mathbb{S}^1)} \leq \varepsilon_0$, then the equation (*) has a unique solution. Moreover, the solution is positive and of $C^{2,\alpha}(\mathbb{S}^1)$.

Return to the logarithmic case

- Existence of weak solution is known, but the origin may lie on the boundary.
- There are examples of f such that the origin touches the boundary of the solution convex bodies: for $n = 2$, parts of the body is described by $(r = \sqrt{x^2 + y^2})$

$$z = r^4 \quad \text{or} \quad z = (r-1)_+^2 \quad \text{(at most } C^{1,1}\text{)}.$$

- Can we find a regular solution for any $f > 0$?

Result 2. Existence of regular solution

Theorem (Choi–Kim–L. in preperation)

Let $f > 0$ be a function in $C^2(\mathbb{S}^n)$. Then the logarithmic Minkowski problem admits a regular ($C^{1,1}$) solution.

Sketch of proof.

1. Consider the following normalized anisotropic Gauss curvature flow

$$X_t = X - f(\nu)K^\alpha \nu.$$

2. Prove diameter estimate $|X| \leq C$ and existence of inner ball
3. Principal curvature estimate $0 < \lambda_1 \leq \lambda_2 \leq C$. □

Dual Minkowski problem

- Rewrite the PDE with $\tilde{q} = n + 1 - q$: In $\mathbb{S}^n$, ($r = \sqrt{u^2 + |\nabla u|^2}$)

$$\det(\nabla_i \nabla_j u + u\delta_{ij}) = \frac{r^{\tilde{q}}}{u} f.$$

- $\tilde{q} > n + 1$: Existence, uniqueness, regularity (Li–Sheng–Wang '20)
- $\tilde{q} < n + 1$: When $f(z) = f(-z)$, existence, uniqueness, and regularity follows.
- If $n = 1$ and $0 < \tilde{q} < n + 1 = 2$, then smooth, positive solution exists for general f (Chen–Li '18).

L_p dual Minkowski problem

- Recall the PDE: In $\mathbb{S}^n$, $(r = \sqrt{u^2 + |\nabla u|^2})$

$$\det(\nabla_i \nabla_j u + u\delta_{ij}) = \frac{r^{\tilde{q}}}{u^{1-p}} f.$$

- $p > q$ $(p + \tilde{q} > n + 1)$: Existence, uniqueness, regularity (Huang–Zhao '18)
- Results for even case when $p > 0, q > 0$; $p < 0, q < 0$; $p > 0, q < 0$.
- Results for general case when $p < q$?

Theorem (Kim–L. 22, arxiv)

Let $p \in (0, 1)$, $q \geq 2$ and let f be a bounded, positive function on $\mathbb{S}^1$. If K is a convex body such that

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then $\|h_K\|_{L^\infty} \leq C$ for some $C = C(p, q, \Lambda)$.

Thank you!