Concentration phenomenon for the Sobolev flow

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① p-Sobolev flow:

$$
(\mathrm{P})\begin{cases}
\partial_t\big(\underbrace{|u|^{q-1}u}\big) - \mathrm{div}\big(\overbrace{\underbrace{|\nabla u|^{p-2}\nabla u}}^{=\Delta_p u}\big) = \lambda(t)|u|^{q-1}u & \text{in } \Omega_\infty = \Omega\times(0,\infty) \\
\|u(t)\|_{q+1} = 1 & t>0 \\
u = 0 & \text{on } \partial\Omega\times(0,\infty) \\
u(0) = u_0 & \text{in } \Omega
\end{cases}
$$

doubly nonlinear

$u = u(x,t)$, $(x,t)\in\Omega_\infty = \Omega\times(0,\infty)$

$\Omega\subset\mathbb{R}^n$, $n\geqq 2$: bdd. domain, $\partial\Omega\in C^\infty$

$1<p<n \quad p\leqq q+1\leqq p^* = \dfrac{np}{n-p}$

ID u_0 : $u_0\in W_0^{1,p}(\Omega)$

$0 < u_0 \leqq \|u_0\|_\infty$ in Ω

$\|u_0\|_{q+1} = 1$

②

· $\lambda(t) = \|\nabla u(t)\|_p^p$: Lag. multi.

(∵) $(\mathrm{P})_1\times u$: $\int_\Omega dx$

$$\Rightarrow \frac{q}{q+1}\underbrace{\frac{d}{dt}\|u(t)\|_{q+1}^{q+1}}_{=0} + \|\nabla u(t)\|_p^p = \lambda(t)\underbrace{\|u(t)\|_{q+1}^{q+1}}_{=1}$$

$(\mathrm{P})_1$:

- · $\partial_t(|u|^{q-1}u) - \Delta_p u = 0$, $q+1>p$: fast diffusion
  $\Rightarrow$ finite time extinc.
- · ODE : $\partial_t(|u|^{q-1}u) = \lambda(t)|u|^{q-1}u$
  $\Rightarrow |u(t)|^{q-1}u(t) = |u_0|^{q-1}u_0\, e^{\int_0^t\lambda(s)ds}$

$\rightsquigarrow$ GE : expected

cf. $p=2$, $\Omega = M$: cpt. smooth mfd.

$\rightsquigarrow$ Yamabe flow $(\mathrm{P})_1$ $(\mathrm{P})_2$

$\Rightarrow (\partial_t - \Delta)R = f(R) \geqq R^2$

R : scalar curv.

③ · gradient flow

$E(u)=\frac{1}{p}\|\nabla u\|_p^p$: p-energy

$u\in W_0^{1,p}(\Omega)$, $\underbrace{\|u\|_{q+1}}_{=\|u\|}=1$

$\varphi\in C_0^\infty(\Omega)$

$$\frac{d}{d\tau}\Big|_{\tau=0} E\left(\frac{u+\tau\varphi}{\|u+\tau\varphi\|}\right)=:\langle\nabla E(u),\varphi\rangle$$

$$=\int\frac{1}{\|u\|^{p-1}}|\nabla u|^{p-2}\nabla u\cdot\left(\frac{1}{\|u\|}\nabla\varphi-\frac{1}{\|u\|^{2+q}}\int|u|^{q-1}u\varphi\,dx\,\nabla\varphi\right)dx$$

$$=\int|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi-\|\nabla u\|_p^p|u|^{q-1}u\varphi\,dx$$

$$=\int\left(-\Delta_p u-\|\nabla u\|_p^p|u|^{q-1}u\right)\varphi\,dx$$

$$\therefore\ \nabla E(u)=-\Delta_p u-\|\nabla u\|_p^p|u|^{q-1}u$$

$(P)_1$: $\frac{d}{dt}\left(|u|^{q-1}u\right)=-\nabla E(u)$

$\{u(t),\ t\ge 0\}\subset W_0^{1,p}(\Omega)\cap\{\|u\|_{q+1}=1\}$

steepest ~~like~~ descent

④ EL : $\nabla E(u)=0$

$$\begin{cases} u\ge 0 : \text{bdd.} \\ -\Delta_p u=\lambda u^q \ \text{in } \Omega \\ u=0 \ \text{on } \partial\Omega \end{cases} \qquad \begin{cases} q+1=p^* \\ \Omega : \text{star-shaped} \end{cases} \Rightarrow u\equiv 0$$

(∵) $-\Delta_p$: Pohzaev Id.
Hopf max.p.
(∵) regularization

· SI : $v\in W_0^{1,p}(\Omega)$, $q+1=p^*$

$\exists\, c_0\|v\|_{q+1}\le\|\nabla v\|_p$, $c_0>0$

$$c_0=\inf_{\substack{v\in W_0^{1,p}(\Omega)\\ \|v\|_{q+1}=1}}\|\nabla v\|_p \qquad \Omega:\text{bdd.}\Rightarrow c_0 : \underline{\text{not}}\ \text{attained}$$

· SI on $\mathbb{R}^n$

$$\begin{cases} v>0 \\ -\Delta_p v=v^q \end{cases}$$

$$v(x)=\left(\frac{a^{\frac{p}{p-1}}n^{\frac{1}{p-1}}\left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}}}{a^{\frac{p}{p-1}}+|x-x_0|^{\frac{p}{p-1}}}\right)^{\frac{n-p}{p}} : \text{T.F.}$$

$v(x)=v_{a,x_0}(x)$

$w(x)=\lambda^{-\frac{n-p}{p^2}}v_{\lambda a,x_0}(x)$

$\Rightarrow -\Delta_p w=\lambda w^q$

$a\in\mathbb{R}$, $x_0\in\mathbb{R}^n$

T.F. $\Rightarrow$ attains c_0

⑤ $\boxed{Q.}$ • GEW, reg.

• Asy. b. : $u(t) \to ?$ as $t \to \infty$

$\uparrow$

volume (energy) concentration

<u>Th. 1</u> (GEW, Reg)

① $\exists u \in C([0,\infty); L^{q+1}(\Omega)) \cap L^\infty(0,\infty; W_0^{1,p}(\Omega))$: GW

② $\lambda(t) = \|\nabla u(t)\|_p^p$

③ $0 < u \leq \|u_0\| e^{\frac{1}{q}\int_0^T \lambda(t)dt}$ in $\Omega \times (0,T)$, $0<T<\infty$

④ $u, \nabla u$: ℓHC

<u>Pr.</u> • max. princ. ⇒ ③ (∵) regularization on time

Alt-Luckhaus's test

• exp. of pos. (: weak Harnack) ⇒ ④

⊕ reg. for evol. p-Lap.

(∵) vol. const. : $\|v(t)\|_{q+1} = 1$

$$\boxed{\begin{cases} \partial_t u^q - \Delta_p u = \lambda(t) u^q \\ \|u(t)\|_{q+1} = 1 \end{cases}}$$

⑥

<u>Cor.</u> (Energy ineq.s)

① GW u satisfies

$$\|\partial_t u^{\frac{q+1}{2}}\|^2_{L^2(\Omega_\infty)}, \ \sup_{t\geq 0} \|\nabla u(t)\|_p^p \leq \|\nabla u_0\|_p^p$$

② $q \geq 1$

① ⇒ $\|\partial_t u^q\|_{L^1(\Omega_\infty)} \leq C(\|\nabla u_0\|_p^p)$

⇝ Asy. beh.

<u>Pr.</u> $u \geq 0$ ⇝ $\varepsilon > 0$, $(u+\varepsilon)$: approximation

ℓHC, gradℓHC, Schauder

⇒ reg. appr. sol.s

⇝ energy estimate

⑦ $$\boxed{\begin{cases} \partial_t u^q - \Delta_p u = \lambda(t) u^q \\ \|u(t)\|_{q+1} = 1 \end{cases}}$$

Th. 2 (concentration-compactness)

$\frac{2n}{n+2} \leq p < n, \quad q+1 = p^*$

$\{t_k\} : t_k \nearrow \infty \qquad \{r_k\} : r_k \searrow 0$

$\Rightarrow$

$\tau_k \nearrow \infty$ as $k \to \infty$

$^\exists$ seq. $\{\tau_k\}$ dep. on $\{t_k\}$, $^\exists N \in \mathbb{N}$

$^\exists$ N-points $\{x_i\} \subset \Omega$, $i = 1, \cdots, N$

$^\exists$ subseq. $\{r_{ki}\} \subset \{r_k\}$

$^\exists$ seq. $\{L_{ki}\} : L_{ki} \nearrow \infty$ as $k \to \infty$

s.t.

- $u(x, \tau_k) - \sum_{i=1}^{N} L_{ki} \chi_{B(x_i, r_{ki})}(x) w_i\left(L_{ki}^{\frac{q+p}{p}}(x - x_i)\right)$

  $\longrightarrow u_\infty(x) \quad (S)$ in $W^{1,p} \cap L^{q+1}(\Omega)$

- $\exists\, \overline{\lambda_\infty} > 0 \qquad 0 \leq u_\infty \in W_0^{1,p}(\Omega)$: bdd. WS of

  $-\Delta_p u = \overline{\lambda_\infty} u^q$ in Ω

  $u_\infty, \nabla u_\infty : HC(\overline{\Omega})$

- $^\exists \underset{\vee\!\!\!\!\!\;\; 0}{\lambda_{\infty i}}$, $i = 1, \cdots, N$, $\quad 0 < w_i \in \mathcal{D}^{1,p}(\mathbb{R}^n)$: bdd. WS of

  $-\Delta_p u = \lambda_{\infty i} u^q$ in $\mathbb{R}^n$

  $w_i, \nabla w_i : \ell HC$ in $\mathbb{R}^n$

- $\|u(t_k)\|_{q+1} \longrightarrow \|u_\infty\|_{q+1} + \sum_{i=1}^{N} \|w_i\|_{q+1}$

- $\|\nabla u(t_k)\|_p \longrightarrow \|\nabla u_\infty\|_p + \sum_{i=1}^{N} \|\nabla w_i\|_p$

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$$\boxed{\begin{cases} \partial_t u^q - \Delta_p u = \lambda(t) u^q \\ \|u(t)\|_{q+1} = 1 \end{cases}}$$

Th.3 (ε-strong compactness)

$\frac{2n}{n+2} \leq p < n, \quad q+1 = p^*$

$\{t_k\} : t_k \nearrow \infty$

$\Longrightarrow$

$^\exists$seq. $\{\tau_k\} : \tau_k \nearrow \infty$ as $k \to \infty$
dep. on $\{t_k\}$

$^\exists \varepsilon_0 > 0$

$^\exists \{x_i\} \subset \Omega, \ i = 1, \cdots, N < \infty$

s.t. $\cdot\ 0 < {}^\forall r << 1, \ i = 1, \cdots, N$

$$\varliminf_{k \to \infty} \int_{\tau_k - r^p}^{\tau_k} \|u(t)\|_{L^{q+1}(B(x_i, r))}^{q+1} dt > \varepsilon_0$$

$\cdot\ u(t_k) \longrightarrow u_\infty$ (as in Th. 2)
(S) in $W^{1,p}_{loc}(\Omega \setminus \{x_1, \cdots, x_N\})$

9 Th.4 (Volume and energy concentration)

$\{t_k\} : t_k \nearrow \infty \qquad \{r_k\} : r_k \searrow 0$

$\Longrightarrow$

$^\exists \varepsilon_0 > 0 \qquad {}^\exists\{\tau_k\} : \tau_k \nearrow \infty$ dep. on $\{t_k\}$

$^\exists \{x_i\} \subset \Omega, \ i = 1, \cdots, N$

$^\exists \{L_{ki}\} : L_{ki} \nearrow \infty$ as $k \to \infty, \ i = 1, \cdots, N$

s.t.

1. $$\varliminf_{k \to \infty} \|u(\tau_k)\|_{L^{q+1}(B(x_i, r_k))}^{q+1} \geq \varepsilon_0$$

2. $x_0 = x_i, \ L_k = L_{ki}, \ i = 1, \cdots, N$

$$v_k(x) = \frac{u\left(x_0 + L_k^{\frac{p-(q+1)}{p}} x, \ \tau_k\right)}{L_k}$$

$\longrightarrow {}^\exists w_i(x)$ (as in Th. 2)
(S) in $W^{1,p}_{loc} \cap L^{q+1}_{loc}(\mathbb{R}^n)$

⑩ key lemma for Th.3 and Th.4

Lem.5 (Local bdd.) $1<p<n$, $q+1=p^*$

$u \geq 0$: WS $\boxed{\partial_t u^q - \Delta_p u = \lambda(t) u^q}$

$a_0>0$, $r_0>0$ s.t. $r_0 \|\nabla u_0\|_p^p \leq 1$

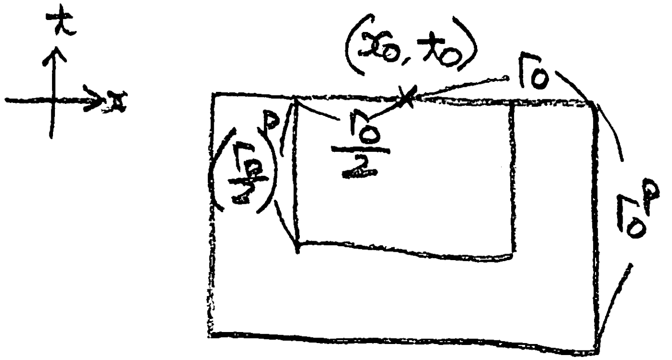
$Q = B(x_0, r_0) \times (t_0 - r_0^p, t_0)$

If
$$\int_{t_0 - r_0^p}^{t_0} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,r_0))} dt \leq a_0$$

$\Rightarrow \exists\, \delta_0 = \delta_0(n,p) \in (0,1)$

s.t.
$$\sup_{Q^-} u \leq \underbrace{8 \left(\frac{r_0}{2}\right)^{-n-p} \delta_0^{-p(1+\frac{2}{n})}}_{=: C_0(r_0, \delta_0)}$$

$Q^- = B(x_0, r_0/2) \times (t_0 - (r_0/2)^p, t_0)$



⑪ Pr. of Th.3

$x_0 \leftarrow \forall x \in \Omega \setminus \{x_i\}_{i=1}^N$, $r_0 \leftarrow r>0$: small

- $\exists \{\tau'_k\} : \tau'_k \nearrow \infty$ s.t. $\displaystyle\int_{\tau'_k - r_0^p}^{\tau'_k} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,r_0))} dt \leq 2\varepsilon_0$
  $\subset \{\tau_k\}$

$a_0 \leftarrow 2\varepsilon_0$ Lem.5 $\Rightarrow \sup_{Q^-} u \leq C_0(r_0, \delta_0)$ $\cdots (*)$

- $\displaystyle\int_\Omega |\partial_t u^q(\tau_k)| dx \to 0$ (∵) $q \geq 1 \Leftrightarrow p \geq \frac{2n}{n+2}$
  $\Rightarrow$ Cor.② : $\|\partial_t u^q\|_{L^1(\Omega_\infty)} \leq$ bdd.
- $\nabla u(\tau_k) \xrightarrow{(S)} {}^\exists \nabla u_\infty$ in $L^p(B(x_0, r_0/2))$ (∵) Cor.$①_2$: energy bdd. ⊕ (*)
- $\lambda(\tau_k) \to {}^\exists \bar{\lambda}_\infty$ (∵) Cor.$①_2$: energy bdd.
- $u(\tau_k) \xrightarrow{(S)} {}^\exists u_\infty$ in $L^r(B(x_0, r_0/2))$, $\forall r \geq 1$
  (∵) Comp. of $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $1 \leq r < p^*$ ⊕ (*)

$\Rightarrow 0 \leq u_\infty \in W_0^{1,p}(\Omega)$: bdd. WS of
$$\begin{cases} -\Delta_p u = \bar{\lambda}_\infty u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(12) **Pr. of Th. 4** $x_0 \leftarrow x_i$

- $\exists \{\tau_k\}: \tau_k \nearrow \infty$ s.t.

$$\|u(\tau_k)\|^{q+1}_{L^{q+1}(B(x_0,r_k))} \ge \varepsilon_0$$

- $\|u(t)\|_{q+1}=1 \Rightarrow \displaystyle\int_{\tau_k-(r_k/2)^p}^{\tau_k} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,\frac{r_k}{2}))}\, dt \le 1$

∴ $a_0 \leftarrow 1$ Lem 5 $\Rightarrow \sup_{Q} u \le \underbrace{8\left(\frac{r_k}{2}\right)^{-n-p}\delta_0^{-p(1+\frac{2}{n})}}_{=:L_k}$

$t_0 \leftarrow \tau_k$

$r_0 \leftarrow r_k$

$$Q = B\left(x_0,\tfrac{r_k}{2}\right)\times\left(\tau_k-\left(\tfrac{r_k}{2}\right)^p, \tau_k\right)$$

Blowing up: $v_k(x,t) = \dfrac{u\Big(\overbrace{x_0 + L_k^{\frac{p-(q+1)}{p}}x}^{=X}, \overbrace{\tau_k+t}^{=T}\Big)}{L_k}$, $\quad \nu_k(t) = \lambda(\tau_k+t)$

$$(x,t)\in \overbrace{B\left(0,\tfrac{r_k}{2}L_k^{\frac{q+1-p}{p}}\right)}^{B_k}\times\left(-\left(\tfrac{r_k}{2}\right)^p, 0\right) \Leftrightarrow (X,T)\in B\left(x_0,\tfrac{r_k}{2}\right)\times\left(\tau_k-\left(\tfrac{r_k}{2}\right)^p,\tau_k\right)$$

$$\frac{r_k}{2}L_k^{\frac{q+1-p}{p}} = c(n,p)\,\delta_0^{-\frac{p\gamma}{n-p}}\, r_k^{\underbrace{1-\frac{p(n+p)}{n-p}}_{<0}} \nearrow \infty$$

$$\gamma = p\left(1+\frac{2}{n}\right)$$

- $\displaystyle\int_{B_k}|v_k(0)|^{q+1}dx = \int_{B(x_0,\frac{r_k}{2})}|u(\tau_k)|^{q+1}dX$

- $\displaystyle\int_{B_k}|\nabla v_k(0)|^{p}dx = \int_{B(x_0,\frac{r_k}{2})}|\nabla u(\tau_k)|^{p}dX$

- $\displaystyle\int_{B_k}\left|\partial_t v_k^{\frac{q+1}{2}}(0)\right|^{2}dx = \int_{B(x_0,\frac{r_k}{2})}\left|\partial_T u(\tau_k)^{\frac{q+1}{2}}\right|^{2}dX$

- $\partial_t v_k^q(0) - \Delta_p v_k(0) = \nu_k(0)\, v_k^q(0)$

● We can take the limit similarly as (13)