Symmetry properties for the Euler equations and semilinear elliptic equations

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Steady Euler equations for an inviscid incompressible fluid

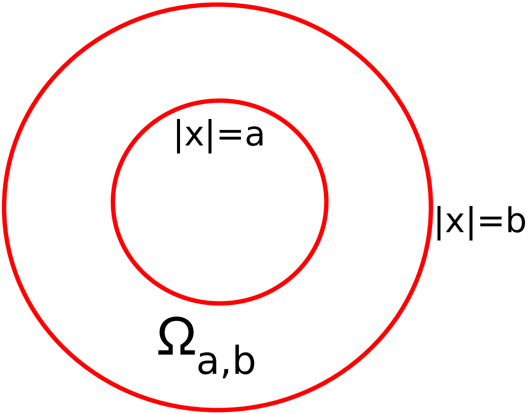
$$\begin{cases} v\cdot\nabla v+\nabla p=0 & \text{in } \Omega \\ \mathrm{div}\ v=0 & \text{in } \Omega \end{cases}$$

with $v \in C^2(\overline{\Omega})$

How does the flow inherit the geometry of the domain ?

- Circular domains $\Longrightarrow$ circular flows ?
- Parallel domains $\Longrightarrow$ parallel flows ?

Sufficient conditions in dimension 2 ?



$$e_r(x) = \frac{x}{|x|}, \quad e_\theta(x) = e_r(x)^\perp$$

Theorem

Assume that $v \cdot e_r = 0$ on $\partial\Omega_{a,b}$ and $|v| > 0$ in $\overline{\Omega_{a,b}}$.

Then v is a circular flow

$$v(x) = V(|x|)\, e_\theta(x)$$

with $V \neq 0$ in $[a, b]$.

- The streamlines $\Xi_x = \{\xi_x(t) : t \in I\}$ are concentric circles

$$\dot{\xi}_x(t) = v(\xi_x(t)), \quad \xi_x(0) = x$$

- Equivalent formulation: any non-circular flow must have a stagnation point in $\overline{\Omega_{a,b}}$.

- It is sufficient to assume that the set of stagnation points is properly included in $C_a = \{|x| = a\}$ or in $C_b = \{|x| = b\}$.

Without the assumption $|v| > 0$, the conclusion does not hold in general !

For any classical function u solving

$$\Delta u + f(u) = 0 \text{ in } \Omega_{a,b}$$

with u constant on $\{|x| = a\}$ and on $\{|x| = b\}$, then

$$v = \nabla^{\perp} u$$

obeys the Euler equations (pressure $p = -|\nabla u|^2/2 - F(u)$ and $F' = f$).

If u has critical points, then v has stagnation points.

If u is not radial, then v is not a circular flow.

Example:

$$\begin{cases} -\varphi'' - r^{-1}\varphi' + r^{-2}\varphi = \lambda\,\varphi \text{ and } \varphi > 0 \text{ in } (a, b) \\ \varphi(a) = \varphi(b) = 0 \end{cases}$$

Then $u(x) = \varphi(r)\,\cos(\theta)$ solves $\Delta u + \lambda u = 0$, with 6 critical points in $\overline{\Omega_{a,b}}$

$\Longrightarrow v = \nabla^{\perp} u$ is a non-circular flow with 6 stagnation points

The condition $|v| > 0$ in $\overline{\Omega_{a,b}}$ is obviously not equivalent to being a circular flow !

There are circular flows with stagnation points (besides the trivial flow !)

Example:

$$\begin{cases} -\phi'' - r^{-1}\phi' = \mu\,\phi \ \text{ and } \ \phi > 0 \ \text{ in } (a,b) \\ \phi(a) = \phi(b) = 0 \end{cases}$$

Then $u(x) = \phi(r)$ solves $\Delta u + \mu\, u = 0$, with

$$\{\text{critical points}\} = C_{r^*} = \{|x| = r^*\}$$

for some $a < r^* < b$

$\Longrightarrow v = \nabla^{\perp} u$ is a circular flow with infinitely many stagnation points

Scheme of the proof

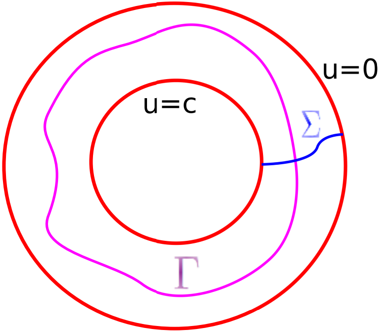
- Stream function u of the flow v:

$$\nabla^{\perp} u = v$$

with

$$\begin{cases} u(x) = c & \text{for } |x| = a \\ u(x) = 0 & \text{for } |x| = b \end{cases}$$

- Any streamline Γ intersects any trajectory Σ of $\dot{\sigma}(t) = \nabla u(\sigma(t))$



- Without loss of generality: $c > 0$
- Then

$$0 < u < c \ \text{ in } \Omega_{a,b}$$

- Vorticity

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \Delta u$$

is constant along the streamlines (from the Euler equations !)

- Semilinear elliptic equation

$$\Delta u + f(u) = 0$$

with $f(s) = -\Delta u\big(\sigma(\theta^{-1}(s))\big)$ for $s \in [0, c]$ and $\theta(t) = u(\sigma(t))$

Theorem [Sirakov]

Let $f : [0, c] \to \mathbb{R}$ be Lipschitz continuous.

Let $\Omega_{a,b} = \{a < |x| < b\} \subset \mathbb{R}^n$ and $u \in C^2(\overline{\Omega_{a,b}})$ solve

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_{a,b} \\ 0 < u < c & \text{in } \Omega_{a,b} \end{cases}$$

with $u = 0$ on $\{|x| = b\}$ and $u = c$ on $\{|x| = a\}$.

Then u is radially symmetric and decreasing:

$$u(x) = U(|x|) \text{ in } \overline{\Omega_{a,b}}$$

and $U'(r) < 0$ for all $a < r < b$.

- Conclusion of the theorem for the Euler equations (with $\Omega_{a,b} \subset \mathbb{R}^2$):

$$v = \nabla^{\perp} u = U'(|x|)\, e_\theta(x)$$

Exterior domains $\Omega_{a,\infty} = \{|x| > a\}$ with $a > 0$

Theorem

Assume that $v \cdot e_r = 0$ on C_a, together with

$$\inf_{\Omega_{a,\infty}} |v| > 0 \quad \text{and} \quad v(x) \cdot e_r(x) = o\Big(\frac{1}{|x|}\Big) \text{ as } |x| \to +\infty.$$

Then v is a circular flow

$$v(x) = V(|x|)\, e_\theta(x)$$

with $V \neq 0$ in $[a, +\infty)$

- The streamlines Ξ_x are concentric circles
- The stream function u is radially symmetric (Liouville-type result)
- The flow is not assumed be bounded, example: $v(x) = |x|\, e_\theta(x)$

Counter-example without the condition $v(x)\cdot e_r(x) = o(1/|x|)$ *as* $|x|\to\infty$:

$$\begin{cases} u = 2\left(\dfrac{r^2}{a^2}-1\right)+\left(\dfrac{r}{a}-\dfrac{a}{r}\right)\cos\theta \\ \qquad\qquad \left(\Delta u = \dfrac{8}{a^2}\ \text{in}\ \Omega_{a,\infty},\quad u=0\ \text{on}\ C_a\right) \\ v = \nabla^{\perp}u = \left[\dfrac{4r}{a^2}+\left(\dfrac{1}{a}+\dfrac{a}{r^2}\right)\cos\theta\right]e_\theta + \left[\left(\dfrac{1}{a}-\dfrac{a}{r^2}\right)\sin\theta\right]e_r \end{cases}$$

One has $\inf_{\Omega_{a,\infty}}|v|\ge 2/a>0$. But

$$v(x)\cdot e_r(x) = \left(\frac{1}{a}-\frac{a}{|x|^2}\right)\frac{x_2}{|x|} \neq o\Big(\frac{1}{|x|}\Big)\ \text{as}\ |x|\to+\infty$$

and v is not a circular flow !

The limiting radial oscillation of the far streamlines is $a/2$.

Scheme of the proof

- Stream function u, with $u = 0$ and $\nabla u \cdot e_r > 0$ on C_a (w.l.o.g.)
- Trajectory of the gradient flow $\dot\sigma(t) = \nabla u(\sigma(t))$ with $\sigma(0) = A \in C_a$

$$|\sigma(t)| \to +\infty \ \text{ and } \ u(\sigma(t)) \to +\infty \ \text{ as } t \overset{<}{\to} T_{max}$$

- For each $t \in [0, T_{max})$, the streamline $\Xi_{\sigma(t)}$ surrounds the origin (continuation argument, with assumption $\inf_{\Omega_{a,\infty}} |v| > 0$)
- All streamlines surround the origin
- $u > 0$ in $\Omega_{a,\infty}$ (and $u(x) \to +\infty$ as $|x| \to +\infty$)
- Radial oscillation $\max_{y \in \Xi_x} |y| - \min_{y \in \Xi_x} |y| \to 0$ as $|x| \to +\infty$
- Equation $\Delta u + f(u) = 0$ in $\Omega_{a,\infty}$ for some C^1 function $f : \mathbb{R}_+ \to \mathbb{R}$
- Method of moving planes (Alexandroff, Gidas-Ni-Nirenberg) $\Longrightarrow$ monotonicity of u in any direction e in $\Omega_{\Xi_x} \cap \Omega_{a,\infty} \cap \{y \cdot e > \varepsilon\}$
- Limiting argument $\Longrightarrow$ radial symmetry of u

Further results in $\Omega_{a,\infty}$ with $\inf_{\Omega_{a,\infty}} |v| > 0$:

$$v \cdot e_\theta > 0 \text{ on } C_a \quad \Longrightarrow \quad \sup_{\Omega_{a,\infty}} \Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big) > 0$$

(argument by contradiction and inversion of variables)

Not true without the condition $\inf_{\Omega_{a,\infty}} |v| > 0$, example:
$v(x) = |x|^{-2} e_\theta(x)$, vorticity $= -|x|^{-3} < 0$

Further results in punctured disks $\Omega_{0,b} = \{0 < |x| < b\}$ with conditions as $|x| \to 0$

Further results in the punctured plane $\Omega_{0,\infty} = \mathbb{R}^2 \backslash \{0\}$ with conditions as $|x| \to 0$ and $|x| \to +\infty$

Serrin-type free boundary problems,
with overdetermined boundary conditions

Theorem

Let Ω be a C^2 non-empty simply connected bounded domain of $\mathbb{R}^2$.
Assume that $v \cdot n = 0$ and $|v|$ is constant on $\partial\Omega$.
Assume moreover that v has a unique stagnation point in $\overline{\Omega}$.

Then, up to a shift,

$$\Omega = B_R$$

and the unique stagnation point of v is the center of the disk.
Furthermore, v is a circular flow:

$$v(x) = V(|x|)\, e_\theta(x) \;\text{ for all } x \in \overline{B_R} \setminus \{0\}$$

with $V \neq 0$ in $(0, R]$ and $V(0) = 0$.

Scheme of the proof

- Stream function u: $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω (w.l.o.g.)
- Equation
$$\Delta u + f(u) = 0 \text{ in } \Omega$$
(because of unique stagnation point z)
- Overdetermined boundary condition
$$\nabla u \cdot n = constant \text{ on } \partial\Omega$$
- If f were Lipschitz continuous on $[0, \max_{\overline{\Omega}} u]$, then Ω is a ball and u is radially symmetric [Serrin]
- Here f can be non-Lipschitz-continuous at the left of $\max_{\overline{\Omega}} u = u(z)$
(example: $v(x) = -4|x|^2 x^\perp$ in $\overline{B_R}$, $u(x) = R^4 - |x|^4$, $\Delta u + 16\sqrt{R^4 - u} = 0$)
- Serrin-type argument in $\Omega \backslash \mathcal{N}(z, \varepsilon) \Longrightarrow$ almost monotonicity of Ω with respect to any line containing z
- $\Longrightarrow \Omega = B(z, R)$ and u is radially symmetric, and v is a circular flow

Related free boundary problems:

- Vorticity $= \mathbf{1}_D$ in $\mathbb{R}^2$ and $v \cdot n = 0$ on ∂D (vortex patch)
  $\Longrightarrow \quad D$ is a disk (Rankine vortex)
- Smooth solutions in $\mathbb{R}^2$ with nonnegative compactly supported vorticity are circular
- Further results for non-stationary uniformly-rotating solutions

[Fraenkel] [Gómez-Serrano, Park, Shi, Yao]

[Hmidi] [Hmidi, Mateu, Verdara] (doubly connected vortex patch)

Conjecture

If D is an open disk, $z \in D$ and $v \in C^2(\overline{D} \setminus \{z\})$ is bounded, and
$v \cdot n = 0$ on ∂D and $|v| > 0$ in $\overline{D} \setminus \{z\}$,
then z is the center of D (and then v is circular)

Theorem

Let ω_1 and ω_2 be two C^2 non-empty simply connected bounded domains of $\mathbb{R}^2$ such that $\overline{\omega_1} \subset \omega_2$, and let

$$\Omega = \omega_2 \setminus \overline{\omega_1}.$$

Assume that $v \cdot n = 0$ and $|v|$ is constant on $\partial\omega_1$ and on $\partial\omega_2$.
Assume moreover that $|v| > 0$ in $\overline{\Omega}$.

Then ω_1 and ω_2 are two concentric disks: up to shift,

$$\Omega = \Omega_{a,b}$$

and v is a circular flow.

- Stream function: $u = c$ on $\partial\omega_1$, $u = 0$ on $\partial\omega_2$ and $0 < u < c$ in Ω
- Equation $\Delta u + f(u) = 0$ in Ω with $f : [0, c] \to \mathbb{R}$ of class $C^1([0, c])$
- Overdetermined conditions: $\nabla u \cdot n$ is constant on $\partial\omega_1$ and on $\partial\omega_2$
- [Reichel] [Sirakov] $\Longrightarrow \Omega = \Omega_{a,b}$ (up to shift)

II. Parallel flows in parallel domains

Parallel flow in dimension N

$$v = (v_1, 0, \cdots, 0)$$

(up to rotation) and

$$v_1 = v_1(x_2, \cdots, x_N)$$

Parallel flow $\Longleftrightarrow$ the pressure p is constant

Parallel flows in two-dimensional domains $\Omega \subset \mathbb{R}^2$?

Two-dimensional strip

$$\Omega_2 = \mathbb{R} \times (0,1) = \big\{x = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_2 < 1\big\}$$

Theorem

Assume that $v_2 = 0$ on $\partial\Omega_2$ ($v \cdot n = 0$ on $\partial\Omega_2$) and

$$\inf_{\Omega_2} |v| > 0.$$

Then v is a parallel flow:

$$v(x_1, x_2) = (v_1(x_2), 0) \ \text{ in } \overline{\Omega_2}.$$

Remark: The flow v is not assumed to be *a priori* bounded in Ω_2. But it is *a posteriori* bounded from the conclusion, since $v = (v_1(x_2), 0)$ and the cross section $[0,1]$ is bounded.

[Kalisch] : additional assumption that $v_1 > 0$ in $\overline{\Omega_2}$

Sufficient condition $\inf_{\Omega_2}|v|>0$*: no stagnation point in* $\overline{\Omega_2}$ *nor at infinity*

- Theorem: any non-parallel flow which is tangential on $\partial\Omega_2$ must have a stagnation point in $\overline{\Omega_2}$ or at infinity.

- Example 1: cellular flow (for $\alpha\neq 0$)

$$\begin{aligned} v(x_1,x_2) &= \nabla^{\perp}\big(\sin(\alpha x_1)\sin(\pi x_2)\big) \\ &= \big(-\pi\sin(\alpha x_1)\cos(\pi x_2),\alpha\cos(\alpha x_1)\sin(\pi x_2)\big) \end{aligned}$$

 with $p(x_1,x_2)=(\pi^2/4)\cos(2\alpha x_1)+(\alpha^2/4)\cos(2\pi x_2)$.

 Stagnation points in $\overline{\Omega_2}$.

- Example 2:

$$v(x_1,x_2)=\nabla^{\perp}\big(\sin(\pi x_2)\,e^{x_1}\big)=\big(-\pi\cos(\pi x_2)\,e^{x_1},\sin(\pi x_2)\,e^{x_1}\big)$$

 with $p(x_1,x_2)=-(\pi^2/2)e^{2x_1}$.

 No stagnation point in $\overline{\Omega_2}$, but $\inf_{\Omega_2}|v|=0$.

But parallel flows $v=(v_1(x_2),0)$ *do not necessarily satisfy* $\inf_{\Omega_2}|v|>0$ *!*

Theorem does not hold in dimension 3

$$\Omega = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ x_2^2 + x_3^2 < 1\right\}$$

Flow

$$v(x) = (1, -x_3, x_2)$$

tangential on the boundary $\partial\Omega$, and

$$1 \le |v| \le \sqrt{2} \text{ in } \Omega.$$

Pressure $p(x) = \dfrac{x_2^2 + x_3^2}{2}$.

The flow is not a parallel flow !

Half-plane

$$\mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty) = \big\{x = (x_1, x_2) \in \mathbb{R}^2,\ x_2 > 0\big\}$$

Theorem

Assume that $v_2 = 0$ on $\partial\mathbb{R}^2_+$ ($v \cdot n = 0$ on $\partial\mathbb{R}^2_+$) and

$$0 < \inf_{\mathbb{R}^2_+} |v| \le \sup_{\mathbb{R}^2_+} |v| < +\infty.$$

Then v is a parallel flow:

$$v(x_1, x_2) = (v_1(x_2), 0) \ \text{ in } \overline{\mathbb{R}^2_+}.$$

The strict inequalities $0 < \inf_{\mathbb{R}^2_+} |v| \le \sup_{\mathbb{R}^2_+} |v| < +\infty$ *cannot be dropped in general*

- Example 1: cellular flow

$$\begin{aligned} v(x_1, x_2) &= \nabla^{\perp}\big(\sin(\alpha x_1)\sin(\pi x_2)\big) \\ &= \big(-\pi\sin(\alpha x_1)\cos(\pi x_2), \alpha\cos(\alpha x_1)\sin(\pi x_2)\big) \end{aligned}$$

  It is bounded in $\mathbb{R}^2_+$, tangential on $\partial\mathbb{R}^2_+$.

  But $\inf_{\mathbb{R}^2_+} |v| = \min_{\mathbb{R}^2_+} |v| = 0$, and v is not a parallel flow.

- Example 2:

$$v(x_1, x_2) = \nabla^{\perp}\big(x_2\cosh(x_1)\big) = \big(-\cosh(x_1), x_2\sinh(x_1)\big)$$

  with $p(x_1, x_2) = -\cosh(2x_1)/4 + x_2^2/2$.

  The flow v is tangential on $\partial\mathbb{R}^2_+$ and $\inf_{\mathbb{R}^2_+} |v| > 0$.

  But $\sup_{\mathbb{R}^2_+} |v| = +\infty$, and v is not a parallel flow.

Open question:

Can the assumption $0 < \inf_{\mathbb{R}^2_+} |v| \le \sup_{\mathbb{R}^2_+} |v| < +\infty$ be replaced with

$$\forall A > 0, \quad 0 < \inf_{\mathbb{R}\times(0,A)} |v| \le \sup_{\mathbb{R}\times(0,A)} |v| < +\infty \ ?$$

The plane $\mathbb{R}^2$

Theorem

Assume that

$$0 < \inf_{\mathbb{R}^2} |v| \leq \sup_{\mathbb{R}^2} |v| < +\infty.$$

Then v is a parallel flow: there exist a unit vector e and $V : \mathbb{R} \to \mathbb{R}$ s.t.

$$v(x) = V(x \cdot e^{\perp})\, e \ \text{ in } \mathbb{R}^2$$

(hence, $v \cdot e$ has a constant sign).

Corollary

Let v be a $C^2(\mathbb{R}^2)$ periodic flow.

If $|v| > 0$ in $\mathbb{R}^2$, then v is a parallel flow.

Corollary

Let v be a parallel flow such that $0 < \inf_{\mathbb{R}^2}|v| \leq \sup_{\mathbb{R}^2}|v| < +\infty$.

If $\|v' - v\|_{L^\infty(\mathbb{R}^2)} \ll 1$ and v' is $C^2(\mathbb{R}^2)$, then v' is a parallel flow.

Remark: in the theorem, if one also assumes that $v \cdot e > 0$ in $\mathbb{R}^2$ for some unit vector e, then the proof is much easier!

Proof in the case of the two-dimensional strip $\Omega_2=\mathbb{R}\times(0,1)$

- Stream function $u\in C^3(\overline{\Omega_2})$ defined by

$$\frac{\partial u}{\partial x_1}=v_2,\quad \frac{\partial u}{\partial x_2}=-v_1$$

$$|\nabla u|=|v|\geq\eta>0$$

Normalization $u(0,0)=0$

$v_2=0$ on $\partial\Omega_2 \Longrightarrow$

$$u=0 \text{ on } \{x_2=0\} \text{ and } u=c \text{ on } \{x_2=1\}\ (c\in\mathbb{R})$$

Each level curve Γ_z of u (connected component of the level set of u containing z) is the streamline of the flow v containing z

- The streamlines are unbounded:

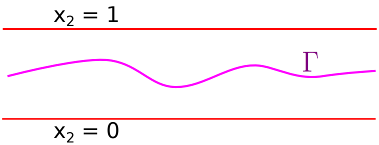
Lemma

Let $\Gamma \subset \overline{\Omega_2}$ be a streamline.

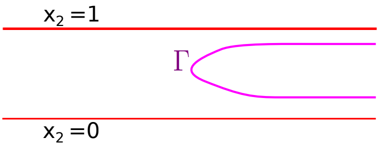
Let $\gamma : \mathbb{R} \to \Gamma$ be a parametrization of Γ.

Then

$$|\gamma(t)| \to +\infty \text{ as } t \to \pm\infty.$$



(d) A possible streamline Γ



(e) Another possible streamline Γ

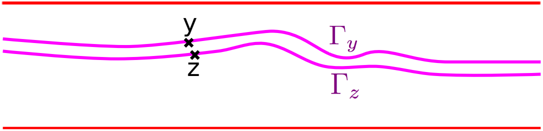
- $B(x,r) = \{y \in \overline{\Omega_2},\ |y-x| < r\}$

Lemma

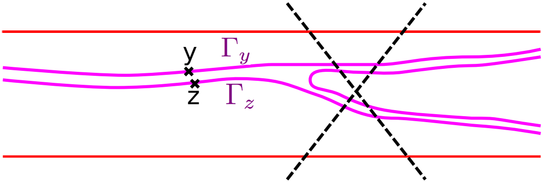
Given point $x \in \overline{\Omega_2}$ and given $\varepsilon > 0$.

Then, there is $r > 0$ such that:

$$\forall\, y, z \in B(x,r), \quad \mathrm{dist}_{\mathcal{H}}(\Gamma_y, \Gamma_z) \le \varepsilon.$$



(f) Two streamlines with $y \simeq z$



(g) Impossible

- Streamlines go from $-\infty$ to $+\infty$ in the direction x_1

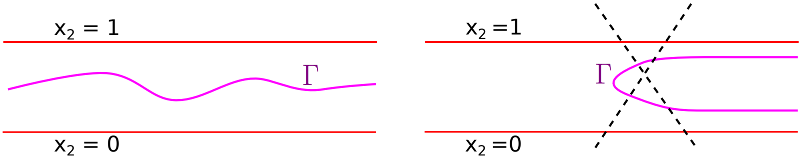
Lemma

Let $\Gamma \subset \overline{\Omega_2}$ be a streamline.

Then Γ has a parametrization $\gamma : \mathbb{R} \to \Gamma$, $t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t))$ such that

$$\gamma_1(t) \to \pm\infty \text{ as } t \to \pm\infty.$$

Proof: continuation argument



(h) Any streamline Γ (i) Impossible

- $v_2 = 0$ on $\partial\Omega_2 \Longrightarrow u = 0$ on $\{x_2 = 0\}$ and $u = c$ on $\{x_2 = 1\}$

  Assume $\dfrac{\partial u}{\partial x_2}(0,0) = -v_1(0,0) > 0$ (w.l.o.g.)

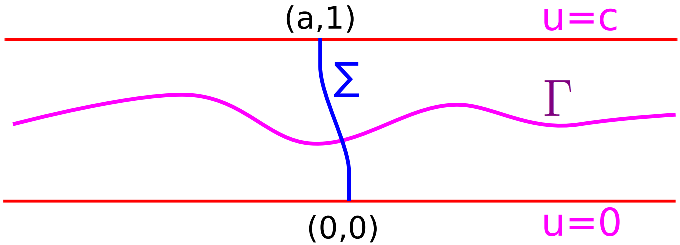
Lemma

The function u is bounded in Ω_2.

Furthermore, $c > 0$ and

$$0 < u < c \text{ in } \Omega_2$$

Trajectory Σ of the gradient flow $\dot\sigma(t) = \nabla u(\sigma(t))$ with $\sigma(0)=(0,0)$ and $t \in [0,\tau]$.



- Vorticity

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \Delta u$$

is constant along the streamlines:

$$v \cdot \nabla(\Delta u) = 0 \text{ in } \overline{\Omega_2}$$

- Semilinear elliptic equation

$$\Delta u + f(u) = 0 \text{ in } \overline{\Omega_2}$$

with $f(s) = -\Delta u\big(\sigma(\theta^{-1}(s))\big)$ for $s \in [0, c]$ and $\theta(t) = u(\sigma(t))$
$(\Delta u(\sigma(t)) + f(u(\sigma(t)) = 0)$

- [Berestycki, Caffarelli, Nirenberg] $\Longrightarrow \dfrac{\partial u}{\partial x_2} \geq 0$
- Liouville-type theorem with sliding method $\Longrightarrow$

$u(x_1, x_2) = U(x_2)$ and $v(x_1, x_2) = (-U'(x_2), 0)$ is a parallel flow.

Proof in the case of the half-plane $\mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty)$

- Potential function $u \in C^3(\overline{\mathbb{R}^2_+})$ defined by

$$\frac{\partial u}{\partial x_1} = v_2, \quad \frac{\partial u}{\partial x_2} = -v_1$$

  Normalization $u(0,0) = 0$

  $v_2 = 0$ on $\partial\mathbb{R}^2_+ \Longrightarrow u = 0$ on $\partial\mathbb{R}^2_+ = \{x_2 = 0\}$

- The streamlines are unbounded.

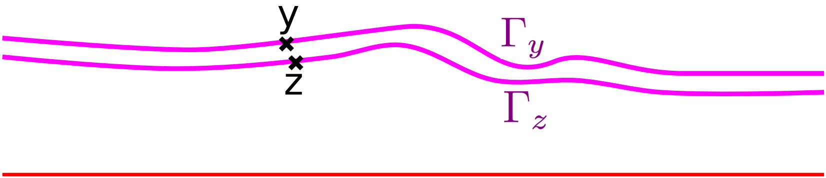
  Let $\gamma : \mathbb{R} \to \Gamma$ be a parametrization of a streamline $\Gamma \subset \overline{\mathbb{R}^2_+}$. Then

$$|\gamma(t)| \to +\infty \text{ as } t \to \pm\infty.$$

- $B(x,r)=\{y\in\overline{\mathbb{R}^2_+},\ |y-x|<r\}$

For any point $x\in\overline{\mathbb{R}^2_+}$ and any $\varepsilon>0$, there is $r>0$ such that:

$$\forall\, y,\, z\in B(x,r),\quad \mathrm{dist}_{\mathcal{H}}(\Gamma_y,\Gamma_z)\le\varepsilon.$$



- All streamlines are bounded in the direction x_2:

Lemma

Let $\Gamma \subset \overline{\mathbb{R}^2_+}$ be a streamline.

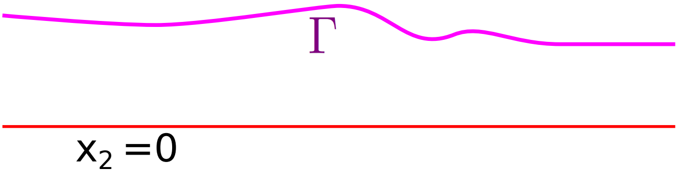
Then there is $A > 0$ such that

$$\Gamma \subset \mathbb{R} \times [0, A]$$

and Γ has a parametrization $\gamma : \mathbb{R} \to \Gamma$, $t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t))$ such that

$$\gamma_1(t) \to \pm\infty \text{ as } t \to \pm\infty.$$

Proof: continuation argument



- $v_2 = 0$ on $\partial\mathbb{R}^2_+ \Longrightarrow u = 0$ on $\{x_2 = 0\}$

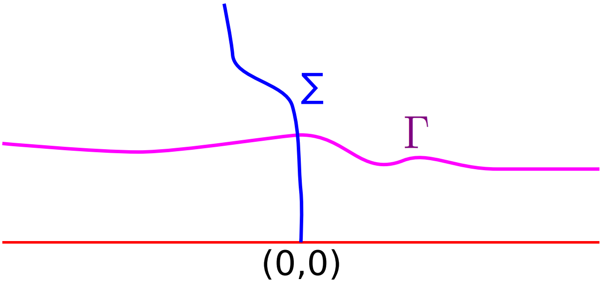
  Assume $\dfrac{\partial u}{\partial x_2}(0,0) = -v_1(0,0) > 0$ (wlog)

Lemma

Then

$$u > 0 \text{ in } \mathbb{R}^2_+$$

Trajectory Σ of the gradient flow $\dot\sigma(t) = \nabla u(\sigma(t))$ with $\sigma(0)=(0,0)$ and $t \in [0,+\infty)$.



- Vorticity $\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} = \Delta u$ constant along the streamlines:

- Semilinear elliptic equation

$$\Delta u + f(u) = 0 \text{ in } \overline{\mathbb{R}^2_+}$$

with $f(s) = -\Delta u\big(\sigma(\theta^{-1}(s))\big)$ for $s \in [0,+\infty)$ and $\theta(t) = u(\sigma(t))$ $(\Delta u(\sigma(t)) + f(u(\sigma(t)) = 0)$

- $u = 0$ on $\partial\mathbb{R}^2_+$ and $u > 0$ in $\mathbb{R}^2_+ \Longrightarrow$

$$\frac{\partial u}{\partial x_2} > 0 \text{ in } \mathbb{R}^2_+$$

[Berestycki, Caffarelli, Nirenberg], [Dancer], [Farina, Sciunzi]

- $|\nabla u| = |v|$ bounded $\Longrightarrow$

$$u(x_1, x_2) = U(x_2)$$

[Farina, Valdinoci]

Conclusion: $v(x_1, x_2) = (-U'(x_2), 0)$ is a parallel flow.

Proof in the case of the plane $\mathbb{R}^2$

- Stream function $u \in C^3(\mathbb{R}^2)$ defined by

$$\frac{\partial u}{\partial x_1} = v_2, \quad \frac{\partial u}{\partial x_2} = -v_1$$

- The streamlines are unbounded.

- The trajectories of the gradient flow are unbounded.

- Each level set of u has only one connected component.

- Equation for the stream function:

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^2.$$

- Argument ϕ of v:

$$\frac{v(x)}{|v(x)|} = (\cos\phi(x), \sin\phi(x)).$$

- Uniformly elliptic equation

$$\boxed{\operatorname{div}(|v|^2\nabla\phi) = 0}$$

Key-estimate

$$|\phi(x)| = O(\ln|x|) \quad \text{as } |x| \to +\infty.$$

- [Moser] $\Longrightarrow$ ϕ is constant $\Longrightarrow$ v is a parallel flow.

Some references

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