The atomic structure of ancient grain boundaries

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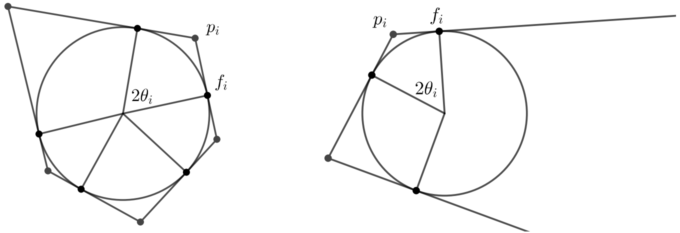
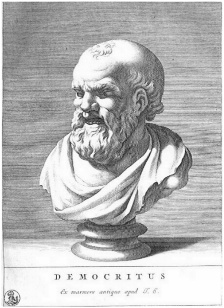


*All original work is joint with Theodora Bourni (UTK) and Giuseppe Tinaglia (KCL).

Democritean atomism

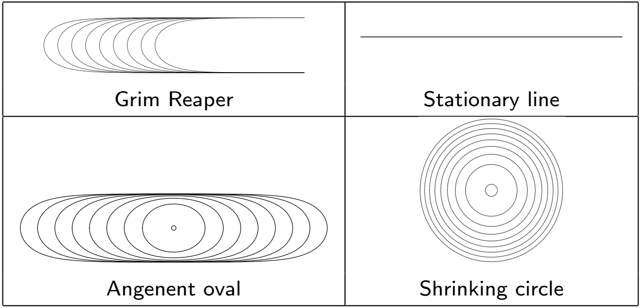
Democritus and the early atomists (Leucippus, Epicurus) held that

- The material cause of all things that exist is the coming together of "atoms" and "void".
- Atoms are eternal and indivisible.
- Atoms can cluster together to create things that are perceivable.
- Differences in shape, arrangement, and position of atoms produce different phenomena.



We will present an atomistic picture of ancient mean curvature flows with the **Grim Reaper** as the fundamental building block.

Convex ancient curve shortening flows



Theorem [DASKALOPOULOS–HAMILTON–ŠEŠUM, BOURNI-L.TINAGLIA, X.-J. WANG]
These are the only convex ancient solutions to curve shortening flow.

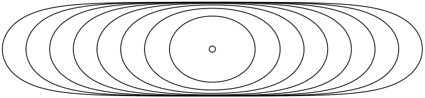
The shrinking circle is entire (it sweeps out all of space).

The Grim Reaper and Angenent oval sweep-out slab regions.

A deep theorem of X.-J, WANG states that that non-entire ancient MCFs necessarily lie in slabs.

The Angenent oval

The Angenent oval is formed from two Grim Reapers in a very specific way.



$$\mathrm{A}_t \doteqdot \{(x, y) \in \mathbb{R}^2 : \cos x = \mathrm{e}^t \cosh y\}$$

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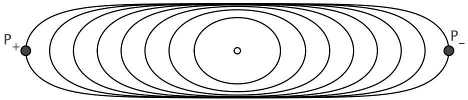


$$\mathrm{A}_t \doteqdot \{(x,y) \in \mathbb{R}^2 : \cos x = \mathrm{e}^t \cosh y\}$$

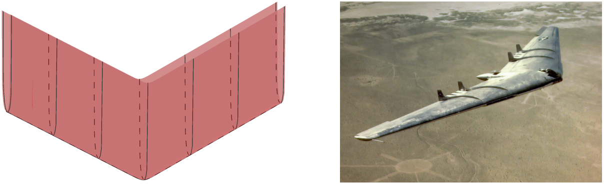
The asymptotic Grim Reapers

$$\mathrm{G}_t^+ \doteqdot \lim_{s\to-\infty}(\mathrm{A}_{t+s} - P_+(s)) \text{ and } \mathrm{G}_t^- \doteqdot \lim_{s\to-\infty}(\mathrm{A}_{t+s} - P_-(s))$$

move with the same speed and thus have the same scale.

Such a configuration is **not** allowed.

Flying wings



A “flying wing” translator flying alongside a Northrop “flying wing” aircraft.

Theorem [Bourni-L.Tinaglia, Hoffman et al., Spruck–Xiao, X.-J. Wang] *The bowl solitons, Grim planes and flying wings are the only non-flat convex translating mean curvature flows in* $\mathbb{R}^3$*.*

For each $\theta \in (0, \frac{\pi}{2})$, there is a convex translator W^θ defined in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^2$ which moves vertically with speed $\sec\theta$ and is asymptotic to two Grim planes (of width π) which make the same angle θ with $\mathbb{R}^2$.

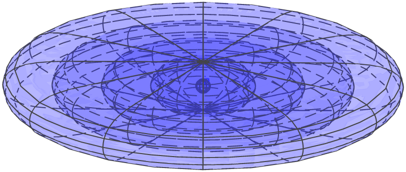
Again, only examples with asymptotic Grim planes of the correct width (equivalently, vertical speed) are admissible.

*Yi Lai recently posted on arXiv a beautiful construction of an analogous family of **steady Ricci solitons**.

Higher dimensions

There is an analogous family of $O(1) \times O(n-1)$-invariant flying wings in $\mathbb{R}^{n+1}$ for each $n \geq 3$. They are the only $O(1) \times O(n-1)$-invariant examples [Bourni-L.-Tinaglia, Hoffman et al. X.-J. Wang].

There is also an $O(1) \times O(n)$-invariant analogue of the Angenent oval in $\mathbb{R}^{n+1}$ for each $n \geq 2$ (the "ancient pancake"). It is the only $O(n)$-invariant example [Bourni-L.-Tinaglia, X.-J. Wang].



The "radius" of the ancient pancake is $r(t) = -t + (n-1)\log(-t) + c_n + o(1)$.

Once again, only examples with asymptotic Grim hyperplanes of the correct width are found.

The ancient pancake is a very useful "barrier", and will play a major role in what follows.

Consequences of the differential Harnack inequality

The second major tool we need is the differential Harnack inequality for ancient solutions (with bounded curvature in each timeslice*).

It immediately implies that:

– the family of support functions $\sigma(\cdot, t) : S^n \to \mathbb{R}$ of $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$ is concave with respect to t,

– $H_*(z) \doteqdot \lim_{s\to-\infty} H(z, s) < \infty$ exists for each normal direction z,

– $\sigma_*(z) \doteqdot \lim_{s\to-\infty} \frac{1}{-s}\sigma(z, s)$ exists for each z,

– $\sigma_*(z) = H_*(z)$,

– $\mathcal{M}_* \doteqdot \lim_{s\to-\infty} \frac{1}{-s}\Omega_s$ exists, where $\partial\Omega_t = \mathcal{M}_t$, and

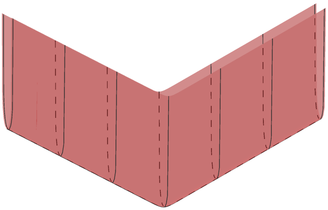
– σ_* is the support function of $\mathcal{M}_*$.

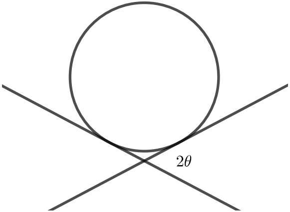
We refer to $\mathcal{M}_*$ as the **squashdown** of $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$.

*We henceforth make the assumption $\sup_{\mathcal{M}_t} H < \infty$ whenever $\mathcal{M}_t$ is noncompact.

The squashdown

Ancient solution	Squashdown
Angenent oval	the interval $[-1,1]$
Grim Reaper	halfline $[-1,\infty) \times \{0\}$
Ancient pancake	unit disk $\overline{B}_1$
Grim hyperplane	halfspace $\{X : \langle X, e_1\rangle \geq -1\}$
Flying wings	circumscribed cone





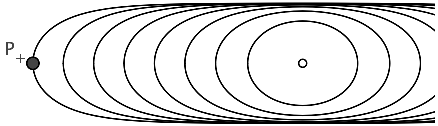
Asymptotic translators

The differential Harnack inequality also ensures that the spacetime translated flows

$$\mathcal{M}_t^j \doteqdot \mathcal{M}_{t+s_j} - X(z_j, s_j)\,, \quad s_j \to -\infty,$$

subconverge to a translator with bulk velocity $\vec{v}$ satisfying

$$-\langle \vec{v}, z \rangle = H_*(z).$$



In particular, $H_*(z) > 0$ whenever $z \neq \pm e_1$.

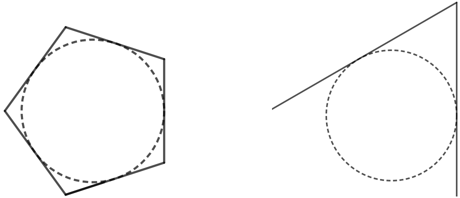
Examples out of regular polytopes

Theorem [Bourni-L.-Tinaglia] *There exists a convex ancient* MCF $\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *in* $\mathbb{R}^{n+1}$ *with* $\mathcal{M}_*^P = P$ *for every regular polytope* $P \subset \mathbb{R}^n$.

$\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *is reflection symmetric across the hyperplane* $\{x = 0\}$ *and inherits the symmetries of* P.

**The asymptotic translators of* $\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *are related to* P *in the obvious way.*

If P *is unbounded, then* $\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *evolves by translation.*



*This is far from immediate: two halfspaces with normals z and w will support the same face of $\mathcal{M}_*$ if $|X(z,t) - X(w,t)| \le o(-t)$ as $t \to -\infty$. But they will only support the same asymptotic translator if $|X(z,t) - X(w,t)| \le O(1)$ as $t \to -\infty$.

Examples out of irregular polytopes

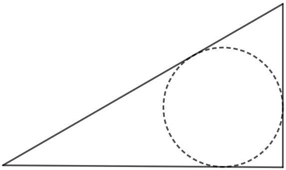
We also obtain examples out of (some) irregular polytopes.

Theorem [Bourni-L.-Tinaglia] *There exists a convex ancient* MCF $\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *in* $\mathbb{R}^{n+1}$ *with* $\mathcal{M}_*^P = P$ *for every simplex* $P\subset\mathbb{R}^n$*.*

$\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *is reflection symmetric across the hyperplane* $\{x=0\}$ *but admits no further symmetries unless* P *does.*

The asymptotic translators of $\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *are related to* P *in the obvious way.*

If P *is unbounded, then* $\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ *evolves by translation.*

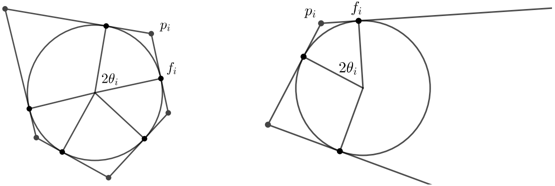


The old-but-not-ancient solutions

The basic idea is to take a limit of *old-but-not-ancient* solutions obtained by flowing suitable configurations of Grim hyperplanes:

Consider a circumscribed polytope $P \subset \mathbb{R}^n$, i.e. a convex set of the form

$$P = \bigcap_{f \in F} \{X \in \mathbb{R}^n : \langle X, z_f \rangle \leq 1\}, \quad F \subset S^{n-1} \text{ finite.}$$

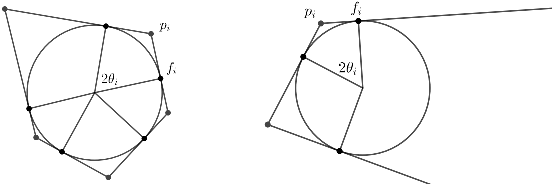


For each $R > 0$, consider the boundary $\mathcal{M}^R$ of the convex body

$$\Omega^R \doteqdot \bigcap_{f \in F} (\Omega_f + R z_f),$$

where Ω_f is the convex region bounded by the Grim hyperplane in $\mathbb{R}^{n+1}$ which passes through the origin and translates in direction $-z_f$.

The old-but-not-ancient solutions



General nonsense yields a convex solution to MCF which is smooth and locally uniformly convex at interior times and $C^{1,1}$ up to the initial time. (When P is unbounded, we use a "doubling argument" [KOTSCHWAR].)

Using the initial Grim planes as outer barriers and the ancient pancake as an inner barrier, we find that

$$\lim_{R\to\infty} \frac{T_R^0}{R} = 1\,,$$

where T_R^0 is the time that the flow reaches the origin.

Denote by $\{\mathcal{M}_t^R\}_{[\alpha_R,\omega_R)}$ the old-but-not-ancient solution obtained by time-translating so that the origin is reached at time 0.

A faux Harnack inequality

By construction, the initial configuration satisfies

$$H_R(z) \geq \langle z, v \rangle$$

for each vertex $v \in V$ of P.

Since both sides are Jacobi fields, the inequality is preserved under the flow. So we obtain the "Harnack" inequality

$$H_R(z, t) \geq \max_{v \in V} \langle z, v \rangle = \sigma_P(z)\,,$$

where σ_P is the support function of P.

Integrating then yields

$$-\frac{\sigma_R(z, t) - \sigma_R(z, s)}{t - s} \geq \sigma_P(z)\,.$$

This ensures that the squashdown of the limit (if it exists) contains P.

A width estimate

In order to obtain a limit, we need a uniform (in R) lower bound for the inradius of $\mathcal{M}_t^R$.

The initial configuration satisfies, by construction,

$$|w_R| \geq \frac{\pi}{2}(1 - H_R)\,, \qquad (\dagger)$$

where $w_R \doteqdot \langle X, e_1 \rangle$. The maximum principle then yields ($\dagger$) for $t > \alpha_R$.

On the other hand, a delicate barrier argument using the ancient pancake and the "Harnack" inequality yields, for any $h \in (0, 1)$

$$\min_{\mathcal{M}_t^R \cap B^n_{C_h}(p)} H_R \leq C_h \mathrm{e}^{-h^2 r}$$

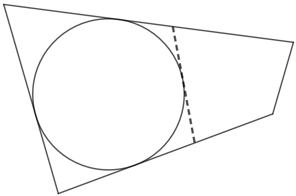
for $r \geq r_h$ and points p which are distance at least $\sim r$ from the "edge" of $\mathcal{M}_t^R$.

The desired inradius lower bound follows.

Taking the limit

General nonsense now yields an ancient solution $\{\mathcal{M}_t^P\}_{t\in(-\infty,\omega)}$ whose squashdown *contains* P.

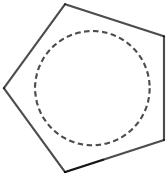
In order to show that $\mathcal{M}_*^P = P$, we need to stop the "faces" from "moving away" as $R \to \infty$.



$\frac{1}{-t}\mathcal{M}_t^R$, $t \sim -\infty$, losing a face as $R \to \infty$.

Preventing faces from wandering off

This can be achieved in some cases via a barrier argument using the ancient pancake $\{\Pi_t\}_{t\in(-\infty,0)}$.



Indeed, since $\{\mathcal{M}_t^R\}_{t\in(\alpha_R,\omega_R)}$ and $\{\Pi_t\}_{t\in(-\infty,0)}$ both reach the origin at time 0, they must intersect for all $t<0$, by the avoidance principle.

With a bit more work, we can deduce that the intersection must happen "near the edge".

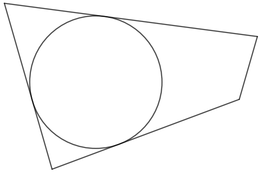
It follows that at least one face of $\mathcal{M}_*$ supports S^{n-1}. In fact, by moving the centre of the pancake, we find that S^{n-1} is *inscribed* in $\mathcal{M}_*$

This suffices to conclude that $\mathcal{M}_* = P$ when P is regular, or a simplex.□

Unique backwards asymptotics

The squashdowns of all these examples circumscribe S^{n-1}.

Theorem [BOURNI-L.-TINAGLIA] *Let* $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$ *be a convex ancient* MCF *in* $\mathbb{R}^{n+1}$*. If* $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$ *is defined in* $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\mathbb{R}^n$*, then its squashdown circumscribes* $\{0\}\times S^{n-1}$*.*



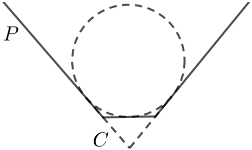
Such configurations are **not** admissible.

Proof. The proof involves some delicate barrier arguments using the ancient pancake and, of course, the differential Harnack estimate. □

We also obtain structure results for the asymptotic translators.

Eternal solutions which do not evolve by translation

Theorem [BOURNI-L.-TINAGLIA] *For each regular, circumscribed cone C, there exists a convex eternal* MCF $\{\mathcal{M}_t\}_{t\in(-\infty,\infty)}$ *to mean curvature flow which sweeps out $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}^n$, is reflection symmetric across the hyperplane $\{0\}\times\mathbb{R}^n$, and whose squash-down is the circumscribed* **truncation** *P of C.*



Since the squashdown of a convex translator is a cone, we conclude that

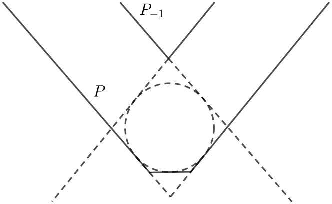
Corollary WHITE*'s conjecture is false: there exist convex eternal solutions to* MCF *which do not evolve by translation.*

Eternal solutions which do not evolve by translation

Consider the old-but-not ancient solution $\{\mathcal{M}_t^R\}_{t\in[0,\omega_R)}$, $\mathcal{M}_t^R = \partial\Omega_t^R$, corresponding to P constructed earlier.

Using barrier arguments and the "Harnack" inequality, we can show that

- $\omega_R = \infty$,
- the limits $\sigma_R^*(z) \doteqdot \lim_{s\to\infty} \frac{1}{s}\sigma_R(z,s)$ and $H_R^*(z) \doteqdot \lim_{s\to\infty} H_R(z,s)$ exist,
- $\sigma_R^*(z) = -H_R^*(z)$, and
- $\mathcal{M}_R^* \doteqdot \lim_{s\to\infty} \frac{1}{s}\Omega_s^R = P_{-1} \doteqdot \bigcap_{z\in F}\{X : \langle X, z\rangle \leq -1\}$.



Eternal solutions which do not evolve by translation

It follows that $H_R(\hat{v}, t)$ increases from 1 at $t = 0$ to $|v|$ as $t \to \infty$, where v is the vertex of C and $\hat{v} \doteqdot v/|v|$.

Using pancake barriers and the "Harnack" inequality, we find that

$$t_{R,\varepsilon} \to \infty \text{ as } R \to \infty\,,$$

where $t_{R,\varepsilon}$ is the first time that $H_R(\hat{v}, \cdot)$ reaches $1 + \varepsilon \in (1, |v|)$.

Now spacetime translate so that $X_R(\hat{v}, 0) = 0$ and $H_R(\hat{v}, 0) = 1 + \varepsilon$, and take $R \to \infty$.

The width estimate implies that the limit is defined in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n$, and hence $\mathcal{M}_*$ circumscribes S^{n-1}.

Since $H(\hat{v}, 0) = 1 + \varepsilon > 1$, the limit cannot be a Grim hyperplane.

The "Harnack" estimate, barrier arguments and the structure of P then imply that $P \subset \mathcal{M}_* \subset C$.

Since $H(\hat{v}, 0) = 1 + \varepsilon < |v|$, $\mathcal{M}_* \subsetneq C$. In particular, $\mathcal{M}_*$ is not a cone.

A little more work yields $\mathcal{M}_* = P$.

Convex eternal solutions

Let $\{\mathcal{M}_t\}_{t\in(-\infty,\infty)}$ be a convex *eternal* MCF.

By the differential Harnack inequality,

- $H^*(z) \doteqdot \lim_{s\to\infty} H(z,s) \in (0,\infty]$ exists for each z,
- $\sigma^*(z) \doteqdot \lim_{s\to\infty} \frac{1}{s}\sigma(z,s) \in [-\infty,0)$ exists for each z,
- $\sigma^*(z) = -H^*(z)$,
- $\mathcal{M}^* \doteqdot \lim_{s\to\infty} \frac{1}{s}\Omega_s$ exists, where $\partial\Omega_t = \mathcal{M}_t$, and
- σ^* is the support function of $\mathcal{M}^*$.

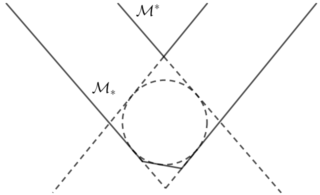
We refer to $\mathcal{M}^*$ as the **forward squashdown** of $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$.

Unique forwards asymptotics

Theorem [Bourni-L.-Tinaglia] *Let* $\{\mathcal{M}_t\}_{t\in(-\infty,\infty)}$ *be a convex* eternal MCF *in* $\mathbb{R}^{n+1}$*. If* $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$ *is defined in* $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}^n$*, then*

$$\mathcal{M}^* = (\mathcal{M}_*)_{-1} \doteqdot \bigcap_{z\in F_*}\{X : \langle X, z\rangle \le -1\},$$

where F_* *is the set of outward normals to* $\mathcal{M}_*$ *which support* S^{n-1}*.*



The main tools in the proof are... barriers and the Harnack inequality! □

Corollary If $\mathcal{M}_*$ is a cone, then $\mathcal{M}_t$ evolves by translation.

Proof: $H(\hat{v}, t)$ is constant in t since it is monotone (by the Harnack ineq.) and $H_*(\hat{v}) = \sigma_*(\hat{v}) = -\sigma^*(\hat{v}) = H^*(\hat{v})$. So the claim follows from the rigidity case of the Harnack inequality. □

We also obtain structure results for the (forwards) asymptotic translators.

Reflection symmetry

Theorem [Bourni-L.-Tinaglia] *Let* $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$ *be a convex ancient* MCF *in* $\mathbb{R}^3$*. If* $\{\mathcal{M}_t\}_{t\in(-\infty,\omega)}$ *is defined in* $(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}^2$*, then it is reflection symmetric across* $\{0\}\times\mathbb{R}^2$.

Thus, our irregular examples admit the smallest possible symmetry groups.

This is proved using a "tilted plane" Alexandrov reflection argument inspired by Korevaar–Kusner–Solomon (cmc surfaces) exploiting the uniqueness of asymptotic translators at faces (Grim planes).

The argument would work in higher dimensions if we had a better understanding of the asymptotic translators (uniqueness of Grim hyperplanes at facets is not enough).